CSE 107 4-16-26

- Thur. 4/23 midterm 1
  - 5:20-6:25 PM exam
  - 6:25-6:30 PM break
  - 6:30-6:55 PM lecture

Ex. let X be r.v. . Let

$Y = aX + b$. Thm $\boxed{E[Y] = aE[X] + b}$

Proof

$$E[Y] = E[aX + b]$$

$$= \sum_x (ax+b) p_X(x)$$

$$= \sum_x \left(ax p_X(x) + b p_X(x)\right)$$

$$= a \cdot \sum_x x p_X(x) + b \cdot \sum_x p_X(x)$$

$$= aE[X] + b$$

∎

note: if $a=0$ then

Recall: Variance

$$\boxed{Var(X) = E[(X-E[X])^2]}$$

Theorem (Variance in terms of moments)

$$\boxed{Var(X) = E[X^2] - (E[X])^2}$$

Proof.

Notation: $E[X] = \mu_X$

SD

$$Var(X) = E\left[(X-\mu_X)^2\right]$$

$$= \sum_x (x-\mu_X)^2 p_X(x)$$

$$= \sum_x (x^2 - 2\mu_X \cdot x + \mu_X^2) p_X(x)$$

$$= \sum_x \left(x^2 p_X(x) - 2\mu_X x p_X(x) + \mu_X^2 p_X(x)\right)$$

$$= \sum_x x^2 p_X(x) - 2\mu_X \cdot \sum_x x p_X(x) + \mu_X^2 \sum_x p_X(x)$$

$$= E[X^2] - 2\mu_X \underbrace{E[X]}_{\mu_X} + \mu_X^2$$

$$= E[X^2] - \mu_X^2$$

$$= E[X^2] - (E[X])^2$$

<u>Ex.</u> (Previous)

$$p_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & \text{if } x = 1 \\ .5 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

<u>Calculated</u>! $E[X] = 1.1$, $E[X^2] = 2.5$

so

$$Var(X) = 2.5 - (1.1)^2 = \ldots = \boxed{1.29}$$

Ex. let $Y = aX + b$. Then

$$\mathrm{Var}(Y) = \mathrm{Var}(aX+b)$$

$$= E[(aX+b)^2] - E[aX+b]^2$$

$$= E[a^2X^2 + 2abX + b^2] - (aE[X]+b)^2$$

$$= a^2E[X^2] + \cancel{2abE[X]} + \cancel{b^2}$$

$$- (a^2E[X]^2 + \cancel{2abE[X]} + \cancel{b^2})$$

$$= a^2(E[X^2] - E[X]^2)$$

$$= a^2\mathrm{Var}(X).$$

Summary

- $E[aX+b] = aE[X] + b$
- $Var(aX+b) = a^2 Var(X)$

Exercise

Prove that

$$E[aX^2+bX+c]$$

$$= aE[X^2]+bE[X]+c$$

note: if $X=$const. then $Var(X)=0$.

Exercise

Prove that if $Var(X)=0$, then $X=$const., i.e. $range(X)=\{\mu_X\}$.

Mean & Variance of common r.v.s

Ex. Bernoulli(p)

$$P_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = 1\cdot p + 0\cdot(1-p) = p$$

$$E[X^2] = 1^2\cdot p + 0^2\cdot(1-p) = p$$

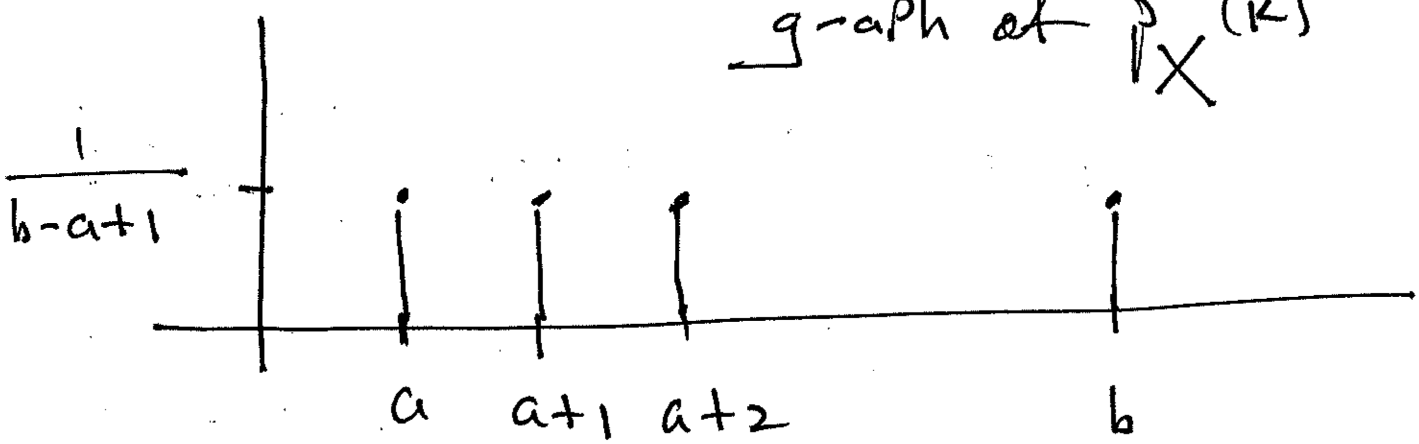
$$Var(X) = E[X^2] - E[X]^2$$

$$= p - p^2 = p(1-p)$$

Ex Uniform(a,b)

Let $a, b \in \mathbb{Z}$, $a < b$. The Discrete uniform r.v. on $\{a, a+1, a+2, \ldots, b\}$ distributes mass evenly:

graph of $p_X(k)$



$$a \le k \le b$$

Thus

$$P_X(k) = \begin{cases} \dfrac{1}{b-a+1} & \text{if } k = a, a+1, \ldots, b \\ 0 & \text{otherwise} \end{cases}$$

so

$$E[X] = \sum_{k=a}^{b} k \cdot \left(\frac{1}{b-a+1}\right)$$

$$= \left(\frac{1}{b-a+1}\right) \cdot \sum_{k=a}^{b} k$$

$$= \left(\frac{1}{b-a+1}\right)\left(\sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k\right)$$

$$= \left(\frac{1}{b-a+1}\right)\left(\frac{b(b+1)}{2} - \frac{a(a-1)}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{b-a+1}\right)\left(\underbrace{(b^2-a^2)}_{(b-a)(b+a)} + (b+a)\right)$$

$$= \frac{1}{2}(b+a)\left(\frac{1}{\cancel{b-a+1}}\right)(\cancel{b-a+1})$$

$$= \boxed{\frac{a+b}{2}}.$$

To compute Var(X), first set

$$Y = X + (1-a)$$

so

$$Var(Y) = Var(X+(1-a)) = Var(X).$$

Let $n = b - a + 1$, and observe Y is uniform on $\{1, 2, \ldots, n\}$

Since

$$X = a \Rightarrow Y = 1$$

$$X = b \Rightarrow Y = n$$

Thus $E[Y] = \frac{n+1}{2}$, and

$$E[Y^2] = \sum_{k=1}^{n} k^2 \cdot \frac{1}{n}$$

$$= \frac{1}{n} \cdot \sum_{k=1}^{n} k^2$$

$$= \frac{1}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6}$$

so

$$Var(Y) = E[Y^2] - E[Y]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= (n+1)\left(\frac{2n+1}{6} - \frac{n+1}{4}\right)$$

$$= \cdots\cdots = \frac{(n+1)(n-1)}{12}$$

$$= \frac{(b-a)(b-a+2)}{12}$$

$$\therefore \boxed{Var(X) = \frac{(b-a)(b-a+2)}{12}}$$

Ex. Poisson(λ)

$$P_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad (k = 0, 1, 2, \ldots)$$

We will show

(1) $E[X] = \lambda$

(2) $E[X^2] = \lambda^2 + \lambda$

Proof of (1)

$$E[X] = \sum_{k=\cancel{0}\,1}^{\infty} k\, e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} \qquad (*)$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \qquad \left\{ \begin{array}{l} \text{replace} \\ k \leftarrow k+1 \end{array} \right.$$

$$= \lambda e^{-\lambda} \cdot \left(\sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!} \right)$$

$$= \lambda \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}}$$

$$= \lambda$$

$\blacksquare$