2.2 Probability Mass Functions

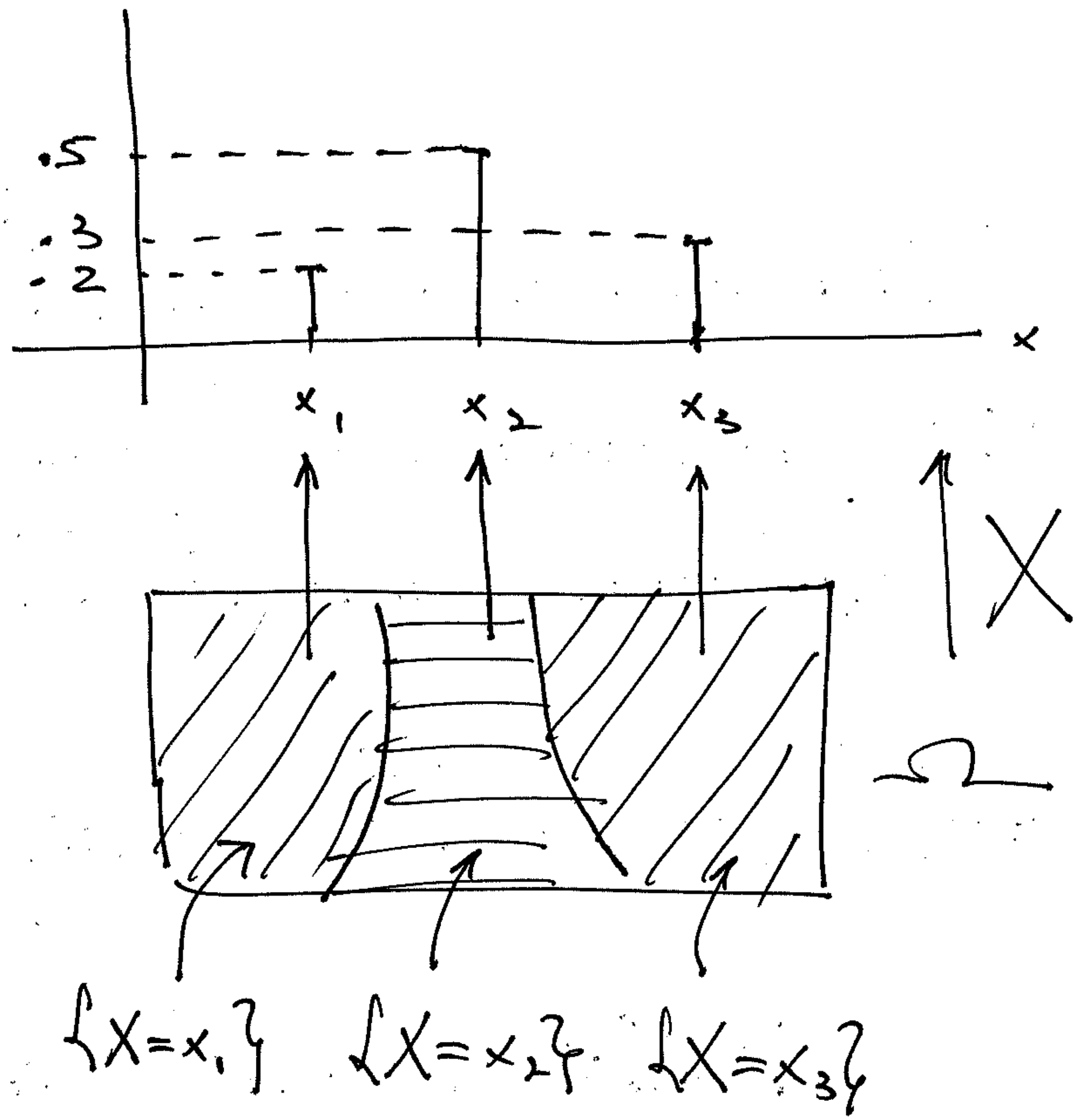
Defn
Let X be a discrete r.v. The Probability Mass Function (PMF) of X is the function

$$p_X(x) = P(\{X = x\}) = P(X = x)$$

observe: $p_X : \mathbb{R} \longrightarrow [0, 1]$

also: $\displaystyle\sum_{x \in \mathrm{range}(X)} p_X(x) = 1$

Ex.



so

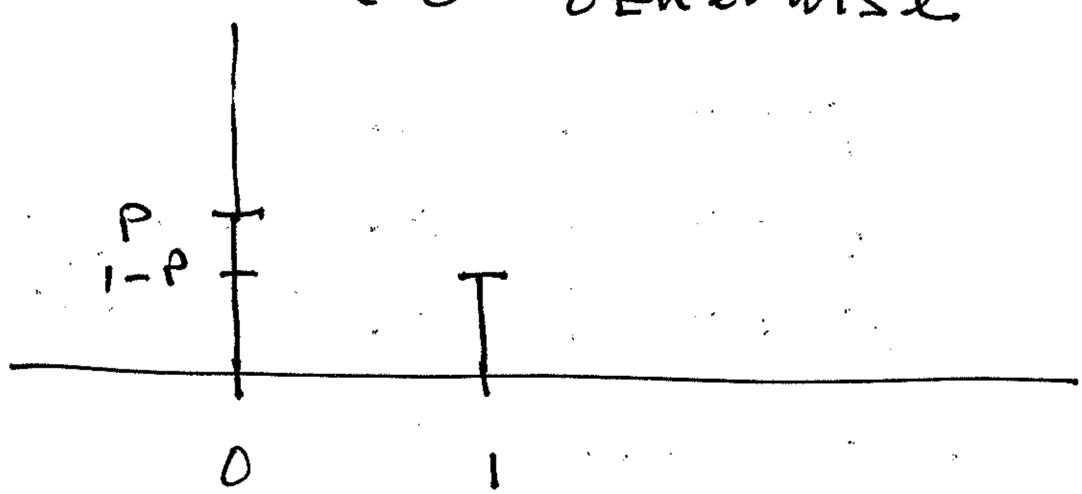
$$p_X(x) = \begin{cases} .2 & \text{if } x = x_1 \\ .5 & \text{if } x = x_2 \\ .3 & \text{if } x = x_3 \\ 0 & \text{otherwise} \end{cases}$$

The Bernoulli Rand. Var.

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \text{ is success} \\ 0 & \text{if } \omega \text{ is failure} \end{cases}$$

its PMF is

$$p_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}$$



CSE 107 1-16-26

Supplemental Lecture

The Binomial r.v.

Perform a sequence of n indep. Bernoulli trials with parameter p. Let

$$X(\omega) = \#\text{ of successes}$$

a bit string of len. n

PMF : Binomial Probabilities

$$P_X(k) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, 1, 2, \ldots, n$.

more specifically

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k = 0, 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

<u>note</u>

$$\sum_{k=0}^{n} P_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \left(p + (1-p) \right)^n = 1^n = 1$$

<u>note</u>: 2 Parameters : n, k

The Geometric r.v.

Perform indep. Bernoulli trials (same param. p) until the 1^{st} success. Let

$X(\omega)$ = length of sequence = # trials

$\uparrow$

bit string of form 000...01

$\boxed{\text{range}(X) = \{1, 2, 3, \ldots\}}$

What is $p_X(k) = P(X=k)$?

$$p_X(k) = (1-p)^{k-1} \cdot p \quad \text{for } k = 1, 2, \ldots$$

more precisely

$$p_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

observe

$$\sum_{K=1}^{\infty} P_X(K) = \sum_{K=1}^{\infty} (1-p)^{K-1} \cdot p$$

$$= p \cdot \sum_{K=1}^{\infty} (1-p)^{K-1}$$

$$= p \cdot \sum_{K=0}^{\infty} (1-p)^{K} \quad \left\{ \begin{array}{l} \text{substitute} \\ K \leftarrow K+1 \end{array} \right.$$

$$= p \cdot \frac{1}{1-(1-p)} = p \cdot \frac{1}{p} = 1$$

<u>note</u>! Geometric has one Parameter, p, Prob. of Success.

The Poisson r.v.

a Poisson r.v. has PMF

$$P_X(k) = \begin{cases} e^{-\lambda} \cdot \frac{\lambda^k}{k!} & \text{if } k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

<u>note</u>: did not specify $X : \Omega \to \mathbb{R}$

<u>observe</u>

$$\sum_{k=0}^{\infty} P_X(k) = \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda} = 1 \checkmark$$

<u>Parameter</u>: $\lambda > 0$

note Poisson(λ) approximates Binomial(n, p), where $\lambda = np$ n is large, p is small.

$$e^{-\lambda} \cdot \frac{\lambda^k}{k!} \approx \binom{n}{k} p^k (1-p)^{n-k}$$

more precisely: let $\lambda > 0$ be constant, $p = \frac{\lambda}{n}$, then

$$\lim_{n \to \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

Ex.

Let $n=200$, $\lambda=2$, $p=\frac{\lambda}{n}=.01$.

Then the prob. of $k=4$ successes in 200 indep. Bernoulli(p) is

$$\binom{200}{4}(.01)^4(.99)^{196} = .\underline{0902}197$$

This is well approximated by

$$e^{-2}\cdot\frac{2^4}{4!} = .\underline{0902}235$$

Discrete Uniform r.v.

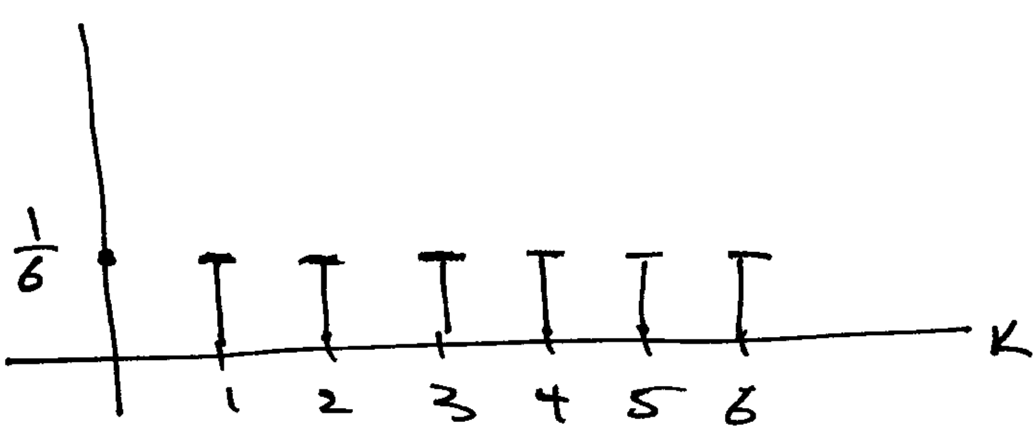
Ex. Roll 1 fair die (6-sided). Let

X = the number showing on die.

so range$(X) = \{1, 2, 3, 4, 5, 6\}$

$$P_X(k) = P(X=k) = \begin{cases} \frac{1}{6} & \text{if } k=1,2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases}$$

graph



In general a discrete uniform r.v. has $\text{range}(X) = \{a, a+1, a+2, \ldots, b\}$ and

$$P_X(k) = \begin{cases} \dfrac{1}{b-a+1} & \text{if } k = a, a+1, \ldots, b \\ 0 & \text{otherwise} \end{cases}$$

<u>note</u>

$$\sum_{k=a}^{b} \left(\frac{1}{b-a+1}\right) = (b-a+1) \cdot \frac{1}{b-a+1} = 1 \quad \checkmark$$

2.3 Functions of a random variable

Given a r.v. X, we may transform X by composition with

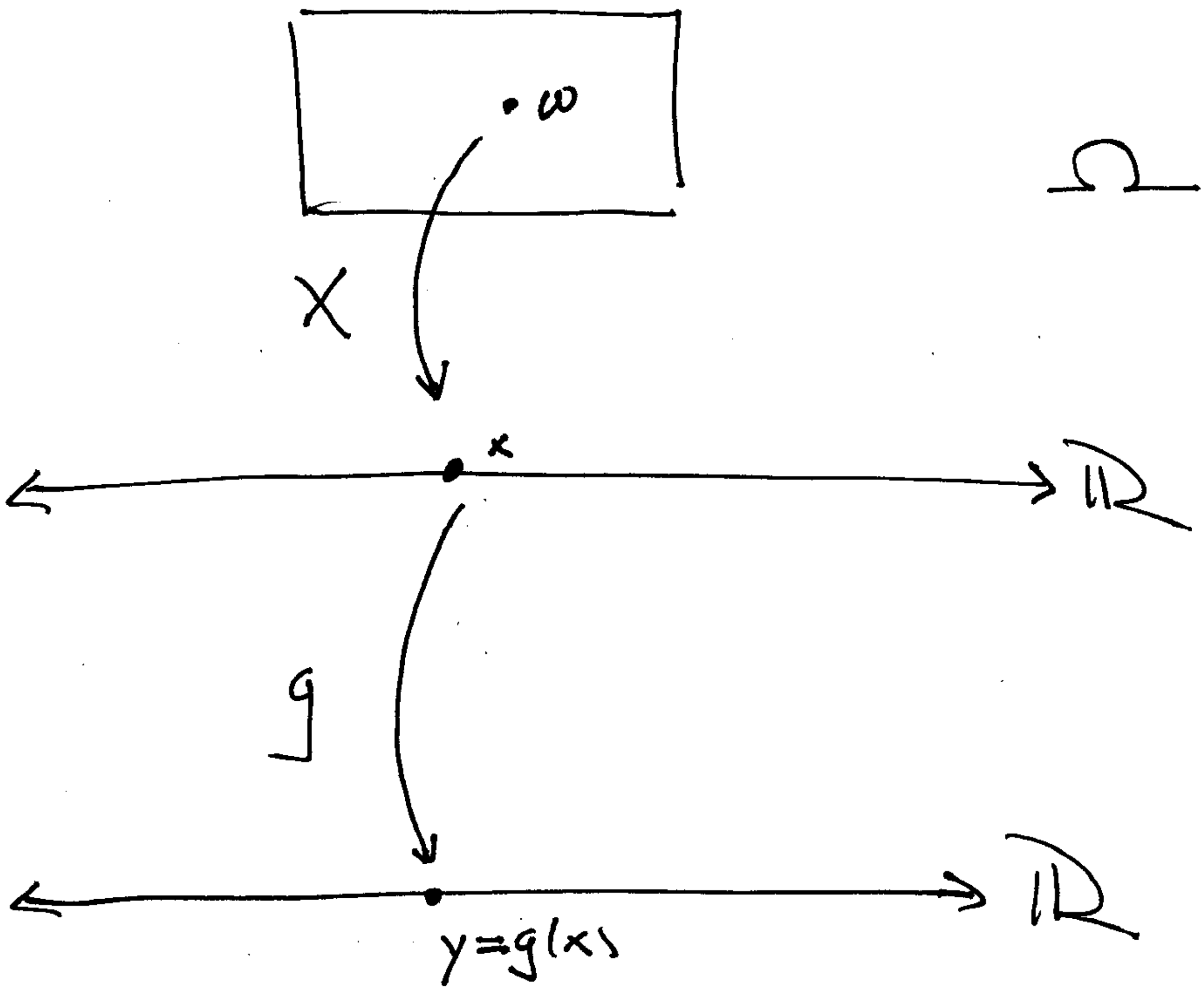
$$g : \mathbb{R} \to \mathbb{R}$$

to obtain $Y = g \circ X = g(X)$.

since

$$X : \Omega \to \mathbb{R}$$

we have $Y = g \circ X : \Omega \to \mathbb{R}$, so Y is itself a new random variable.


ω
Ω
X
x
$\mathbb{R}$
g
$\mathbb{R}$
$y=g(x)$

Ex.

Suppose X has PMF

$$p_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & \text{if } x = 1 \\ .5 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = X^2$. Find PMF of Y.

$(g(x) = x^2)$

$$\begin{aligned} p_Y(y) &= P(Y = y) \\ &= P(X^2 = y) \\ &= P(X = \pm\sqrt{y}) \\ &= P(X = \sqrt{y} \text{ or } X = -\sqrt{y}) \end{aligned}$$

$$= P(X=\sqrt{y}) + P(X=-\sqrt{y})$$

$$= \begin{cases} \cancel{.2 \text{ if } \sqrt{y}=-1} \\ .3 \quad \sqrt{y}=1 \\ .5 \quad \sqrt{y}=2 \end{cases} + \begin{cases} .2 \text{ if } -\sqrt{y}=-1 \\ \cancel{.3 \quad -\sqrt{y}=1} \\ \cancel{.5 \quad -\sqrt{y}=2} \end{cases}$$

$$= \begin{cases} .3 \text{ if } y=1 \\ .5 \text{ if } y=4 \\ 0 \text{ otherwise} \end{cases} + \begin{cases} .2 \text{ if } y=1 \\ 0 \text{ otherwise} \end{cases}$$

$$= \begin{cases} .5 \text{ if } y=1 \\ .5 \text{ if } y=4 \\ 0 \text{ otherwise} \end{cases}$$