CSE 107 3-31-26

1.1 Sets

Read this !!

1.2 Probabalistic Models

A Probability Model consists of

(1) a set Ω called the sample space. The elements

$$\omega \in \Omega$$

are outcomes of an experiment

A subset $A \subseteq \Omega$ is called an <u>event</u>.

(2) A <u>Probability Law</u> P that assigns to an event $A \subseteq \Omega$ a real number $P(A)$, the <u>Probability of A</u>.

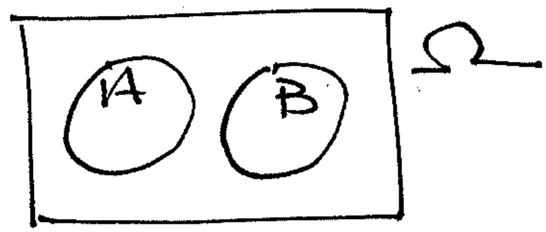
<u>Axioms</u>

(i) $P(A) \geq 0$ for all $A \subseteq \Omega$

(ii) $P(\Omega) = 1$

(iii) <u>Additivity</u>

- finite additivity
  - if $A, B \subseteq \Omega$ are disjoint $(A \cap B) = \emptyset$



then $P(A \cup B) = P(A) + P(B)$

  - if $A_1, A_2, \ldots, A_n \subseteq \Omega$ are pairwise disjoint $(A_i \cap A_j = \emptyset)$,

then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

- countable additivity
  - if $A_1, A_2, A_3, \ldots \subseteq \Omega$ are pairwise disjoint ($A_i \cap A_j = \phi$), then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

<u>Ex.</u> flip a fair coin

$$\Omega = \{H, T\}$$

<u>notation</u>: $H = \{H\}$, $T = \{T\}$

Fair means $P(H) = P(T)$ $*$

Also $\{H\} \cup \{T\} = \{H, T\} = \Omega$, and $\{H\} \cap \{T\} = \phi$, so by axiom (iii)

$$1 = P(\Omega) = P(H) + P(T) \quad **$$

By $*$ and $**$ $P(H) = P(T) = \frac{1}{2}$

In general if $A \subseteq \Omega$, then $A \cup A^c = \Omega$ and $A \cap A^c = \phi$

so $\boxed{P(A) + P(A^c) = 1}$.

But $P(A) \geq 0$, $P(A^c) \geq 0$ by axiom 1. Hence

$\boxed{P(A) \leq 1}$ and $P(A^c) \leq 1$.

Thus

$$0 \leq P(A) \leq 1$$

for any $A \subseteq \Omega$.

Evidently $P : \mathbb{P}(\Omega) \to [0,1]$

Let $A = \Omega$ above. Then

$$
\begin{aligned}
1 &= P(\Omega) \\
&= P(\Omega \cup \Omega^c) \\
&= P(\Omega \cup \phi) \\
&= P(\Omega) + P(\phi) \\
&= 1 + P(\phi)
\end{aligned}
$$

$\therefore P(\phi) = 0$.

<u>Previous example</u>: $\Omega = \{H, T\}$

events : Ω, $\{H\}$, $\{T\}$, ϕ

Prob : 1, $\frac{1}{2}$, $\frac{1}{2}$, 0.

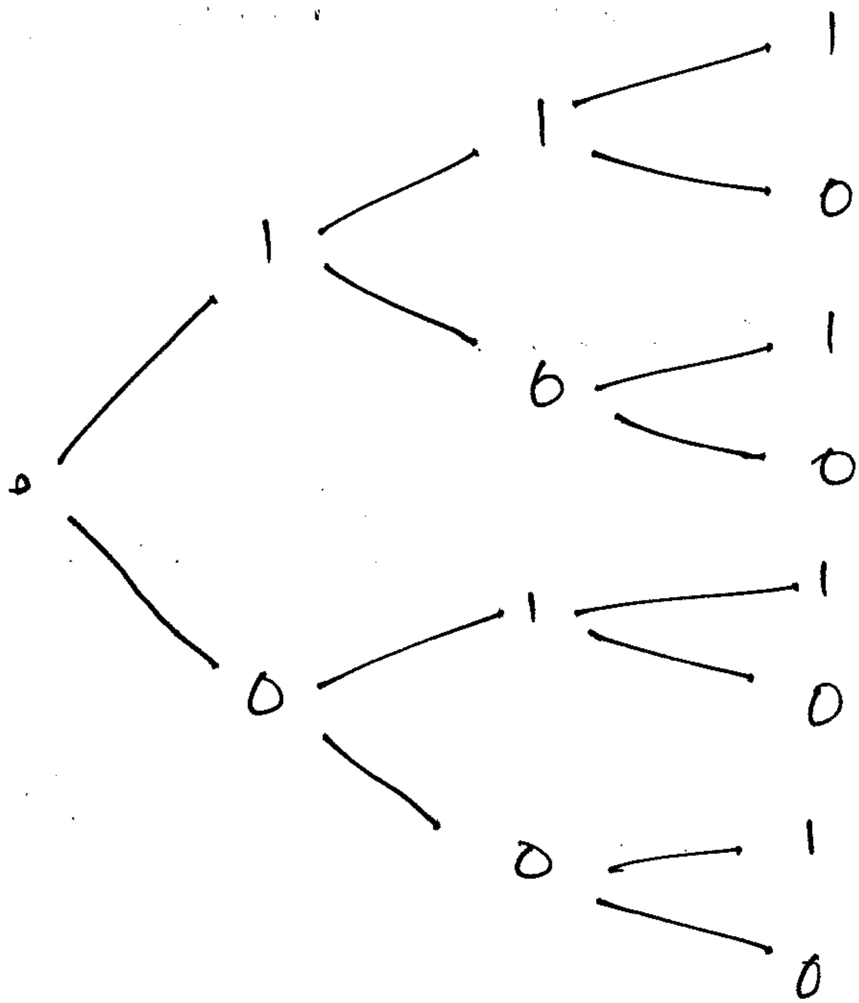
<u>Ex.</u> flip 3 fair coins

<u>notation</u>: $1 = H$, $0 = T$

$\Omega = \{000, 001, 010, 011, 100, 101, 110, 111\}$

'Fair' means outcomes are equally likely.

Tree diagram



Outcomes	Prob
111	1/8
110	1/8
101	1/8
100	
011	
010	
001	
000	1/8

Suppose

$$A = \{\text{exactly 1 head}\}$$
$$= \{100, 010, 001\} = \{100\} \cup \{010\} \cup \{001\}$$

so

$$P(A) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}.$$

In general

Discrete Probability Law

if Ω is finite, then $P(\cdot)$ is determined by probabilities of the singleton events

$$P(\omega) \text{ for } \omega \in \Omega.$$

say $A = \{\omega_1, \omega_2, \ldots, \omega_k\} \subseteq \Omega$.

Then

$$P(A) = P(\omega_1) + P(\omega_2) + \cdots + P(\omega_n).$$

Special case:

Discrete Uniform Probability Law

If Ω is finite as above, and all $\omega \in \Omega$ are equally likely, then

$$P(A) = \frac{|A|}{|\Omega|}$$

Codewords: 'random', 'uniform' mean 'equally likely outcomes.

Ex two 6-sided dice are thrown. The dice are fair in that all outcomes are equally likely

what is Ω ?

Sum= 2 3 4 5 6 7 8 9 10 11 12

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

$P(\text{Sum}=2) = 1/36$
$3) = 2/36$
$4) = 3/36$
$5) = 4/36$
$6) = 5/36$
$7) = 6/36$
$8) = 5/36$
$9) = 4/36$
$10) = 3/36$
$11) = 2/36$
$12) = 1/36$