CSE 107
Probability and Statistics for Engineers
Information for Final Exam

The Law of Iterated Expectations:

$$E[X] = E[E[X|Y]]$$

The Law of Total Variance:

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y])$$

Discrete Convolution Product:

$$(g * h)(n) = \sum_{k} g(k)h(n-k)$$

Continuous Convolution Product:

$$(g * h)(z) = \int_{-\infty}^{\infty} g(x)h(z-x)\, dx$$

Random Variables:

Discrete Uniform(a, b) where $a, b \in \mathbb{Z}$

$$p_X(k) = \begin{cases} \dfrac{1}{b-a+1} & \text{if } a \le k \le b \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{\lfloor x \rfloor - a + 1}{b-a+1} & a \le x < b \\ 1 & x \ge b \end{cases}$$

$$E[X] = \frac{a+b}{2} \qquad \mathrm{Var}(X) = \frac{(b-a)(b-a+2)}{12}$$

Bernoulli(p) with $0 < p < 1$

$$p_X(k) = \begin{cases} p & \text{if } k = 1 \\ 1-p & \text{if } k = 0 \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$E[X] = p \qquad \mathrm{Var}(X) = p(1-p)$$

Binomial(n, p) with $0 < p < 1$ and $n = 0, 1, 2, \ldots$

$$p_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } 0 \le k \le n \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \displaystyle\sum_{k=0}^{\lfloor x \rfloor} \binom{n}{k} p^k (1-p)^{n-k} & 0 \le x < n \\ 1 & x \ge n \end{cases}$$

$$E[X] = np \qquad \mathrm{Var}(X) = np(1-p)$$

Geometric(p) with $0 < p < 1$

$$p_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 - (1-p)^{\lfloor x \rfloor} & x \geq 1 \end{cases}$$

$$E[X] = \frac{1}{p} \qquad \text{Var}(X) = \frac{1-p}{p^2}$$

Poisson(λ) with $\lambda \geq 0$

$$p_X(k) = \begin{cases} e^{-\lambda} \cdot \dfrac{\lambda^k}{k!} & \text{if } k = 0, 1, \dots \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \displaystyle\sum_{k=0}^{\lfloor x \rfloor} e^{-\lambda} \cdot \frac{\lambda^k}{k!} & x \geq 0 \end{cases}$$

$$E[X] = \lambda \qquad \text{Var}(X) = \lambda$$

Continuous Uniform(a, b) where $a, b \in \mathbb{R}$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & a \leq x < b \\ 1 & x \geq b \end{cases}$$

$$E[X] = \frac{a+b}{2} \qquad \text{Var}(X) = \frac{(b-a)^2}{12}$$

Exponential(λ) with $\lambda \geq 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

$$E[X] = \frac{1}{\lambda} \qquad \text{Var}(X) = \frac{1}{\lambda^2}$$

Normal(μ, σ) with $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad F_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\, dt$$

$$E[X] = \mu \qquad \text{Var}(X) = \sigma^2$$

Standard Normal ($\mu = 0$ and $\sigma = 1$)

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt$$

$$E[X] = 0 \qquad \text{Var}(X) = 1$$

Distributions Associated with Bernoulli and Poisson Processes

Pascal Distribution of order k

Let Y_k be the k^{th} arrival time in a Bernoulli process with parameter p. Then $Y_k = T_1 + T_2 + \cdots + T_k$ where T_i are independent geometric random variables with paramenter p, and $k \geq 1$.

$$p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k} \qquad \text{for } t = k, k+1, k+2, \ldots$$

$$E[Y_k] = \frac{k}{p} \qquad \mathrm{Var}(Y_k) = \frac{k(1-p)}{p^2}$$

Erlang Distribution of order k

Let Y_k be the k^{th} arrival time in a Poisson process with parameter λ. Then $Y_k = T_1 + T_2 + \cdots + T_k$ where T_i are independent exponential random variables with paramenter λ, and $k \geq 1$.

$$f_{Y_k}(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!} \qquad \text{for } t \in (0, \infty)$$

$$E[Y_k] = \frac{k}{\lambda} \qquad \mathrm{Var}(Y_k) = \frac{k}{\lambda^2}$$

Equations Associated with Markov Chains

Consider a Markov chain model with state space $S = \{1, 2, \ldots, m\}$ and state transition probabilities p_{ij} for $i, j \in S$. Let X_n denote the state at time $n \geq 0$, and let $r_{ij}(n) = P(X_n = j \mid X_0 = i)$ be the n-step transition probabilities. Then $r_{ij}(n)$ satisfies the following recurrence.

Chapman-Kolmogorov Equations: $$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) \cdot p_{kj} \qquad \text{for all } i, j \in S \text{ and } n \geq 1$$

If a Markov chain consists of one aperiodic recurrent class and possibly some transient states, and π_j denotes the steady-state probabilities ($j \in S$), then π_j satisfies

Balance Equations: $$\pi_j = \sum_{k=1}^{m} \pi_k \cdot p_{kj} \qquad \text{for all } j \in S$$

Normalization Equations: $$1 = \sum_{k=1}^{m} \pi_k$$

Standard Normal Cumulative Distribution Function

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998