CSE 107
Probability and Statistics for Engineers
The Gaussian Integral

The following integrals are used to establish some basic facts about the Normal distribution. The first is often just called the *Gaussian Integral*.

(1) $$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Proof:
Square the left hand side, then switch the double integral to polar coordinates.

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\cdot\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\, rdrd\theta \\
&= \int_{0}^{2\pi} d\theta \cdot \int_{0}^{\infty} re^{-r^2}\, dr \\
&= 2\pi\cdot\left(-\frac{1}{2}\right)e^{-r^2}\Big|_{0}^{\infty} \\
&= -\pi\cdot(0-1) \\
&= \pi
\end{aligned}$$

proving (1). ■

(2) $$\int_{-\infty}^{\infty} e^{-u^2/2}\, du = \sqrt{2\pi}$$

Proof:
Do the substitution $x = u/\sqrt{2}$, to get $x^2 = u^2/2$ and $du = \sqrt{2}\, dx$. Using (1) we have

$$\int_{-\infty}^{\infty} e^{-u^2/2}\, du = \int_{-\infty}^{\infty} e^{-x^2}\sqrt{2}dx = \sqrt{2}\cdot\sqrt{\pi} = \sqrt{2\pi}$$

■

Now let X be a random variable with PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ and σ are real parameters (with $\sigma > 0$). The following identity establishes that this function is a valid PDF.

(3) $$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = 1$$

Proof:
Do the substitution $u = (x-\mu)/\sigma \implies \sigma du = dx$ and apply (2) to get

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-u^2/2}\,\sigma\, du = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = 1$$

■

It remains to show that the parameters μ and σ are the mean and standard deviation of X, respectively. We begin with the variance of X.

(4) $$\mathrm{Var}(X) = \sigma^2$$

from which it follows that the standard deviation of X is σ.

Proof:
Using the same substitution as above gives $y = (x-\mu)/\sigma \implies dx = \sigma\, dy$, and

$$\begin{aligned}
\mathrm{Var}(X) &= E[(X-\mu)^2] \\
&= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x-\mu)^2\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \sigma^2 y^2\, e^{-y^2/2}\,\sigma\, dy \\
&= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2\, e^{-\frac{y^2}{2}}\, dy
\end{aligned}$$

Integration by parts with $u = y, dv = ye^{-y^2/2}dy \implies du = dy, v = -e^{-y^2/2}$ yields

$$\mathrm{Var}(X) = \frac{\sigma^2}{\sqrt{2\pi}} \left(-ye^{-\frac{y^2}{2}}\right)\Bigg|_{-\infty}^{\infty} - \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -e^{-\frac{y^2}{2}}\, dy$$

$$= 0 + \frac{\sigma^2}{\sqrt{2\pi}} \cdot \sqrt{2\pi} \qquad \text{using (2)}$$

$$= \sigma^2$$

as required. ■

To prove $E[X] = \mu$, we use the symmetry of $f_X(x)$ about the vertical line at $x = \mu$. We first prove two lemmas, from which the result will follow.

Lemma 1
If Y is a continuous random variable with $f_Y(y) = f_Y(-y)$ for all $y \in \mathbb{R}$, then $E[Y] = 0$.

Proof:
By definition

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{0} y f_Y(y) dy + \int_{0}^{\infty} y f_Y(y) dy$$

Do the substitution $t = -y$ in the first integral. Then $y = -t, dy = -dt$, and

$$\int_{-\infty}^{0} y f_Y(y) dy = \int_{\infty}^{0} (-t) f_Y(-t)(-1) dt$$

$$= \int_{\infty}^{0} t f_Y(t) dt \qquad \text{using the hypothesis } f_Y(-t) = f_Y(t)$$

$$= -\int_{0}^{\infty} t f_Y(t) dt$$

Thus

$$E[Y] = -\int_{0}^{\infty} t f_Y(t) dt + \int_{0}^{\infty} y f_Y(y) dy = 0$$

as claimed. ■

Lemma 2
If Y is a continuous random variable with $f_Y(c + y) = f_Y(c - y)$ for all $y \in \mathbb{R}$ and some constant c, then $E[Y] = c$.

Proof:
Using the CDF of Y we have

$$F_Y(c + y) = P(Y \leq c + y) = P(Y - c \leq y) = F_{Y-c}(y)$$

Upon differentiating with respect to y, we get $f_Y(c+y) = f_{Y-c}(y)$. Also

$$F_Y(c-y) = P(Y \leq c-y) = P(Y-c \leq -y) = F_{Y-c}(-y)$$

and again, differentiating both sides with respect to y, yeilds $-f_Y(c-y) = -f_{Y-c}(-y)$, whence $f_Y(c-y) = f_{Y-c}(-y)$. Combining these with our hypothesis gives us

$$f_{Y-c}(y) = f_{Y-c}(-y)$$

for all $y \in \mathbb{R}$. Lemma 1 now says that $0 = E[Y-c] = E[Y] - E[c] = E[Y] - c$, from which it follows that $E[Y] = c$, as claimed. ■

Now we return to the random variable X defined by the PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Observe that $f_X(\mu+x) = f_X(\mu-x)$ for any $x \in \mathbb{R}$. Lemma 2 immediately says

(5) $$E[X] = \mu$$

The random variable X with the above PDF is called the Normal or Gaussian random variable, and is sometimes abbreviated simply Normal(μ, σ). Many other important facts about normal distributions will be covered in lecture. See here for a proof that the sum of two Normal random variables is again normal, which uses the convolution product and includes many algebraic details.