

CSE 107

Midterm 2 Review Problems

1. Alice's drive time from work to home is such that in *light traffic*, she takes between 1 hour and 75 minutes, uniformly distributed. In *heavy traffic* her drive time is uniformly distributed between 75 minutes and 95 minutes. Assume that traffic is equally likely to be light or heavy.
 - a. Let X be Alice's commute time, in minutes. Determine the PDF $f_X(x)$.
 - b. Determine Alice's expected commute time.
2. Bob throws a dart at a circular target of radius r . He hits the target with certainty, but is equally likely to hit any point within the target. Let Z be the distance from Bob's dart to the center of the target.
 - a. Find the CDF $F_Z(z)$ and the PDF $f_Z(z)$.
 - b. Find the mean $E[Z]$.
 - c. Find the variance $Var(Z)$.

3. A city's temperature in degrees Celsius is modeled as a normal random variable X with mean 10 and standard deviation 10. Let Y be its temperature in Fahrenheit, where X and Y are related by

$$X = \frac{5(Y - 32)}{9}.$$

What is the probability that the temperature is above 77 degrees Fahrenheit?

4. Let X be a normal random variable with mean μ and standard deviation σ .
 - a. Find the probability that X is within 1.25σ of its mean μ , i.e. find

$$P(\mu - 1.25\sigma \leq X \leq \mu + 1.25\sigma)$$

- b. Determine $c \in \mathbb{R}$ for which the probability that X is within $c\sigma$ of its mean μ is 0.8294, i.e. find c such that

$$P(\mu - c\sigma \leq X \leq \mu + c\sigma) = 0.8294$$

5. Let Y be a normal random variable with variance 1, and with mean another random variable X . Suppose X is continuous uniform on the interval $[1, 3]$.
 - a. Find the PDF $f_Y(y)$.
 - b. Find the conditional PDF $f_{X|Y}(x|y)$.
 - c. Suppose we sample Y and get $Y = 3$. What is the probability that $X \leq 2$?
 - d. Find $E[Y]$.

6. Let X and Y be jointly continuous random variables, and suppose

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & \text{if } 0 < y \leq 1 \text{ and } 1 - y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 2y & \text{if } 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Hint: Before you do the following problems, draw a picture of the region defined by the inequalities $0 < y \leq 1$ and $1 - y \leq x \leq 1$.

- Determine the joint PDF $f_{X,Y}(x,y)$.
 - Determine the marginal PDF $f_X(x)$.
 - Determine the expected value $E[X]$.
 - Determine the conditional expectation $E[X|Y = y]$.
7. Let X be an exponential random variable with parameter λ , and let $Y = X + 1$. Determine the PDF $f_Y(y)$.
8. Alice is at the casino with a choice of two games. The first returns winnings (positive or negative) that are normally distributed with parameters $\mu = 1$ and $\sigma = 2$. The second is uniformly distributed with winnings in the range -1 to 2 . (All amounts are in dollars.) She flips a coin with $P(\text{head}) = p$ to decide which game to play. If heads, she plays the first game, and if tails, she plays the second. Determine her expected winnings, in terms of p .
9. Let X and Y be independent, jointly continuous random variables, where X is uniform on $[a, b]$ and Y is uniform on $[c, d]$.
- Write the PDFs $f_X(x)$ and $f_Y(y)$.
 - Write the CDFs $F_X(x)$ and $F_Y(y)$.
 - Let $Z = \max(X, Y)$. Determine the PDF $f_Z(z)$. (Assume $a \leq d$ and $c \leq b$, so that the two intervals overlap.)
 - Let $Z = X + Y$. Determine the PDF $f_Z(z)$.
10. A point is chosen at random (i.e. uniformly) from within the triangle

$$\{ (x, y) \mid 0 \leq y \leq x \leq 1 \}$$

(Hint: draw this triangle before you go any further.)

- Find the joint PDF $f_{X,Y}(x,y)$
- Find the marginal PDF $f_Y(y)$
- Find the conditional PDF $f_{X|Y}(x|y)$
- Find the conditional expectation $E[X | Y = y]$

11. Let X be a uniform continuous random variable on $[1, 5]$, and let $Y = \ln(X)$. Find the PDF $f_Y(y)$.
(Hint: use the 2-step process for finding a derived distribution.)
12. Let X and Y be exponential random variables with parameters $\lambda_1 = 2$ and $\lambda_2 = 3$ respectively. Assume X and Y are independent. Use the convolution product to find the PDF $f_Z(z)$ of the random variable $Z = X + Y$.
13. Suppose that X and Y are random variables with $\text{Var}(X) = \text{Var}(Y)$. Prove that $X + Y$ and $X - Y$ are uncorrelated.