CSE 107 6-3-25

- Final exam: Mon. 6/9, 7:30-9:30 PM
- SETs: due Sun. 6/8, 11:59 PM

Incentive: If response rate $\geq 75\%$, I will add 0.5% to everyone's overall score.

# 7.1 Discrete Markov Chains

state space

$$S = \{1, 2, \ldots, m\} \subseteq \mathbb{Z}^+$$

let $X_0, X_1, X_2, \ldots$ be a seq. of r.v.s taking values in $S$

$$X_n = \text{state of process at time } n.$$

state transition probabilities:

$$p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i)$$

note: does not depend on $n$.

Markov Property:

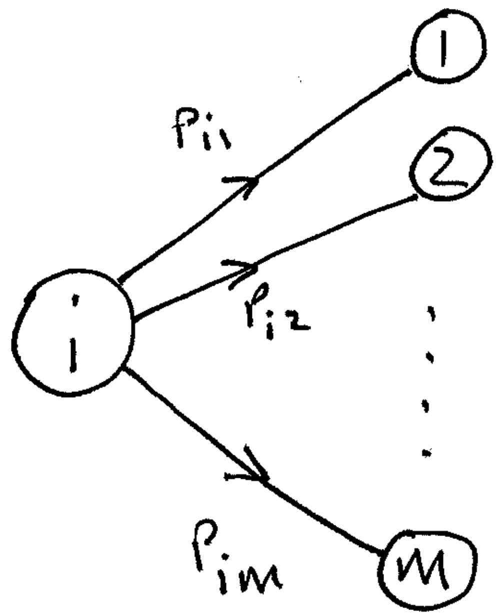
$$P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0)$$

$$= P(X_{n+1}=j \mid X_n=i) = p_{ij}$$

- State space $S$
- transition Prob. $p_{ij}$
- Markov Property

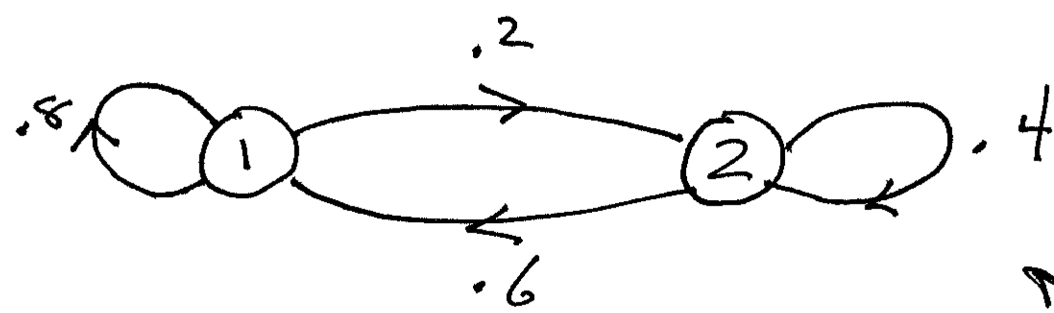
$\Bigg\}$ defines a Markov Chain Model.

<u>note</u>: $\sum_{j=1}^{m} p_{ij} = 1$ (for all $i \in S$)




<u>Ex</u>. $m = 2$, $S = \{1, 2\}$

$p_{11} = .8$, $p_{12} = .2$, $p_{21} = .6$, $p_{22} = .4$




State Transition diagram

Suppose $X_0 = 1$.

- Find $P(X_2 = 2)$

$$P(X_2 = 2) = P(X_2 = 2 \mid X_1 = 1) \cdot P(X_1 = 1)$$
$$+ P(X_2 = 2 \mid X_1 = 2) \cdot P(X_1 = 2)$$
$$= p_{12} \cdot P(X_1 = 1) + p_{22} \cdot P(X_1 = 2)$$

note

$$P(X_1 = 1) = P(X_1 = 1 \mid X_0 = 1) \cdot \overbrace{P(X_0 = 1)}^{1}$$
$$+ P(X_1 = 1 \mid X_0 = 2) \cdot \underbrace{P(X_0 = 2)}_{0}$$
$$= p_{11}$$

also $P(X_1 = 2) = p_{12}$

Thus if $X_0 = 1$, then

$$P(X_2 = 2) = P_{12} \cdot P_{11} + P_{22} \cdot P_{12}$$

$$= P_{11} \cdot P_{12} + P_{12} \cdot P_{22} = \boxed{.24}$$

Similarly, if $X_0 = 1$, then

$$P(X_2 = 1) = P_{11}^2 + P_{12} P_{21} = \boxed{.76}$$

In the same manner, if $X_0 = 2$, then

$$P(X_2 = 1) = P_{21} P_{11} + P_{22} P_{21} = \boxed{.72}$$

and

$$P(X_2 = 2) = P_{21} P_{12} + P_{22}^2 = \boxed{.28}$$

# Transition Probability Matrix

$$R = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix}$$

observe

$$R^2 = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

$$= \begin{pmatrix} P_{11}^2 + P_{12}P_{21} & P_{11}P_{12} + P_{12}P_{22} \\ P_{21}P_{11} + P_{22}P_{21} & P_{21}P_{12} + P_{22}^2 \end{pmatrix}$$

$$= \begin{pmatrix} .76 & .24 \\ .72 & .28 \end{pmatrix}$$

so $ij^{th}$ entry in $R^2$ is Prob. of going from $i$ to $j$ in 2 steps.

## Defn

A <u>Probability Matrix</u> is a square $m \times m$ matrix in which each row sums to 1, i.e. if $A = (a_{ij})$ $(1 \le i \le m, 1 \le j \le m)$, then

$$\sum_{j=1}^{m} a_{ij} = 1 \quad (\text{for } 1 \le i \le m)$$

## Exercise

Prove that the product of two Probability matricies is another Prob. matrix.

Defn

The $n$-step transition probabilities

are

$$r_{ij}(n) = P(X_n = j \mid X_0 = i)$$

[note: same as $P(X_{n+k} = j \mid X_k = i)$]

for any $i, j \in S$, and $n \geq 0$.

observe: $r_{ij}(1) = p_{ij}$

also note:- $r_{ij}(0) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

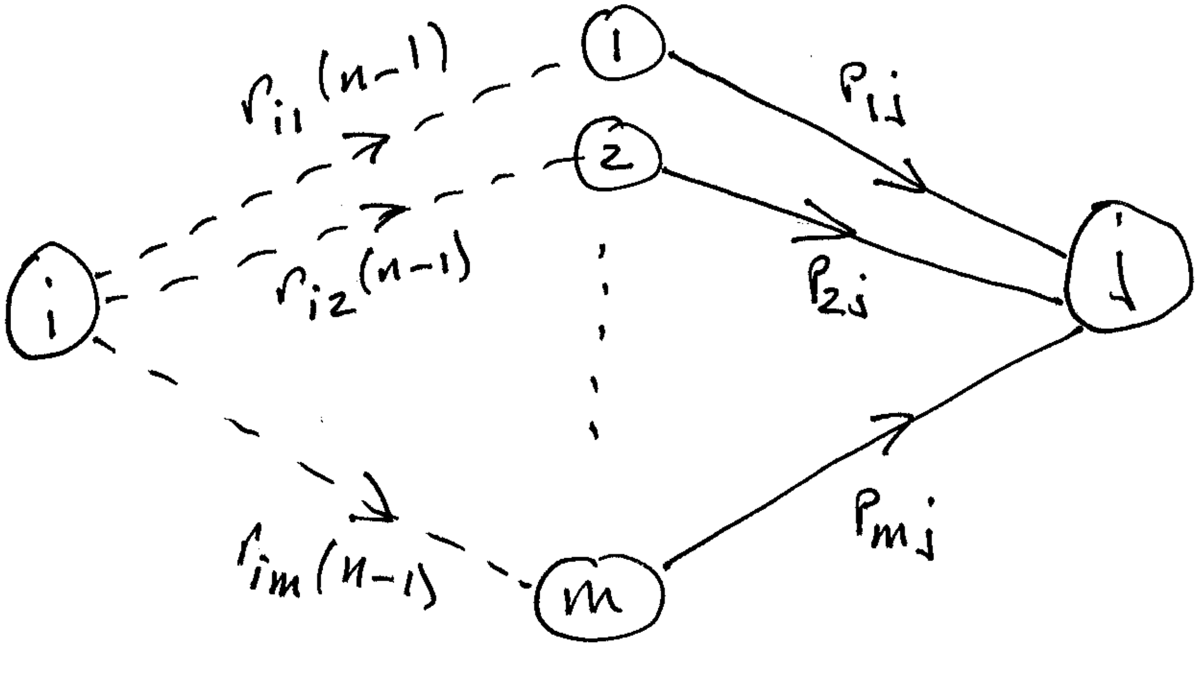
## Theorem (Chapman-Kolmogorov)

$$\boxed{r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)\cdot p_{kj}} \quad (i,j \in S,\ n \geq 1)$$

## matrix version

$$\boxed{R^{n} = R^{n-1}\cdot R}$$

## Picture



Ex. Previous

$$R^3 = \begin{pmatrix} .76 & .24 \\ .72 & .28 \end{pmatrix} \begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix} = \begin{pmatrix} .752 & .248 \\ .744 & .256 \end{pmatrix}$$

compute $R^{10}$ !!

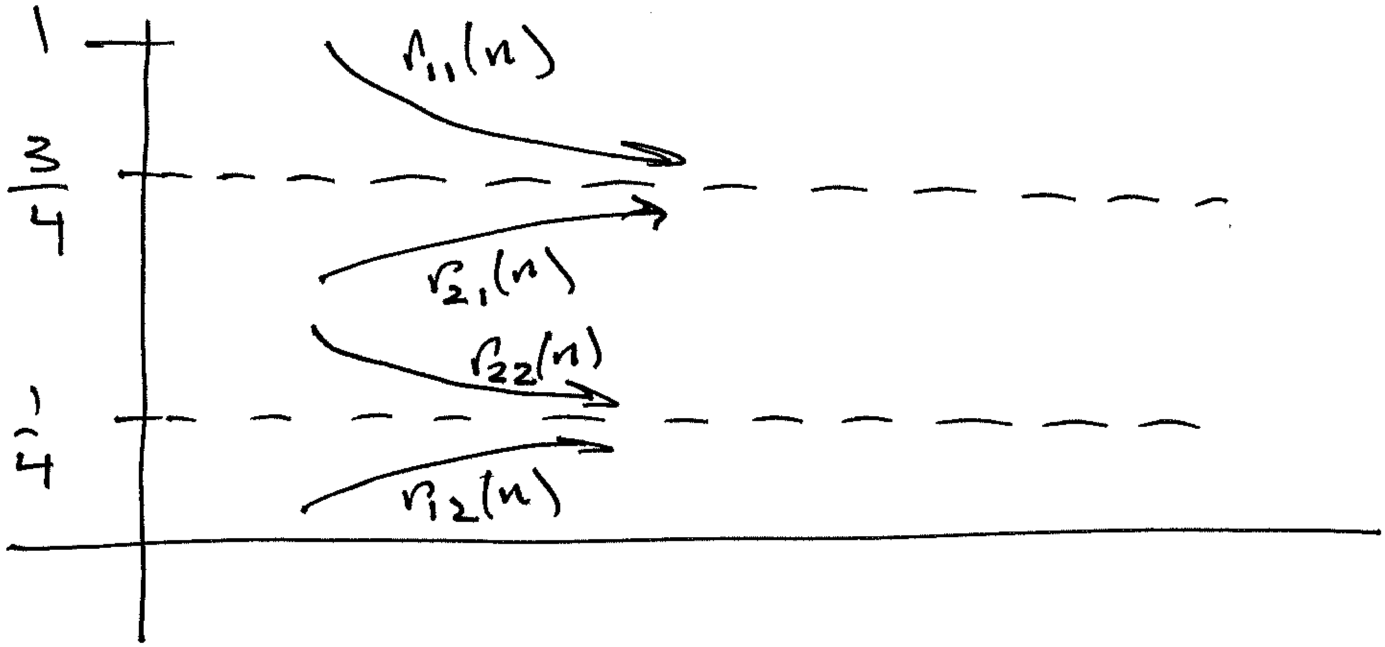
It can be shown that

$$\lim_{n \to \infty} R^n = \begin{pmatrix} .75 & .25 \\ .75 & .25 \end{pmatrix}$$

i.e.

$$r_{11}(n) \to \frac{3}{4}^{+}, \quad r_{12}(n) \to \frac{1}{4}^{-}$$

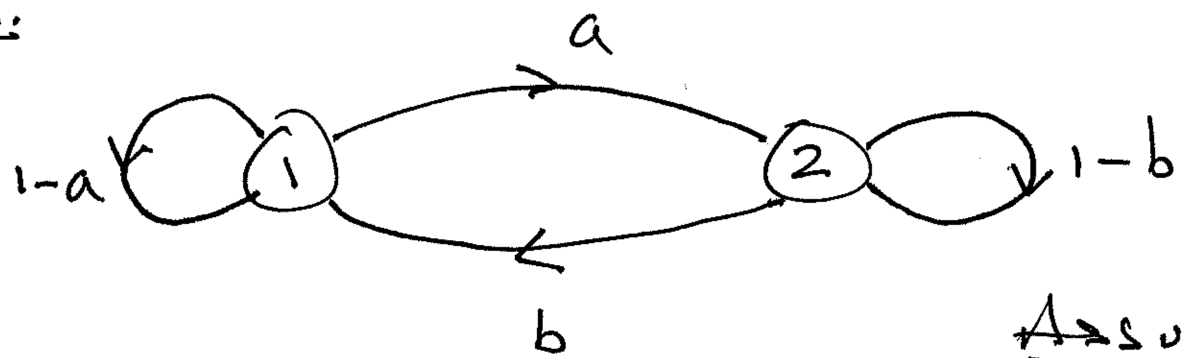
$$r_{21}(n) \to \frac{3}{4}^{-}, \quad r_{22}(n) \to \frac{1}{4}^{+}$$




Thus

$$P(X_n = 1) \approx \frac{3}{4}, \quad P(X_n = 2) \approx \frac{1}{4}$$

Ex.




Assume:

$$R = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

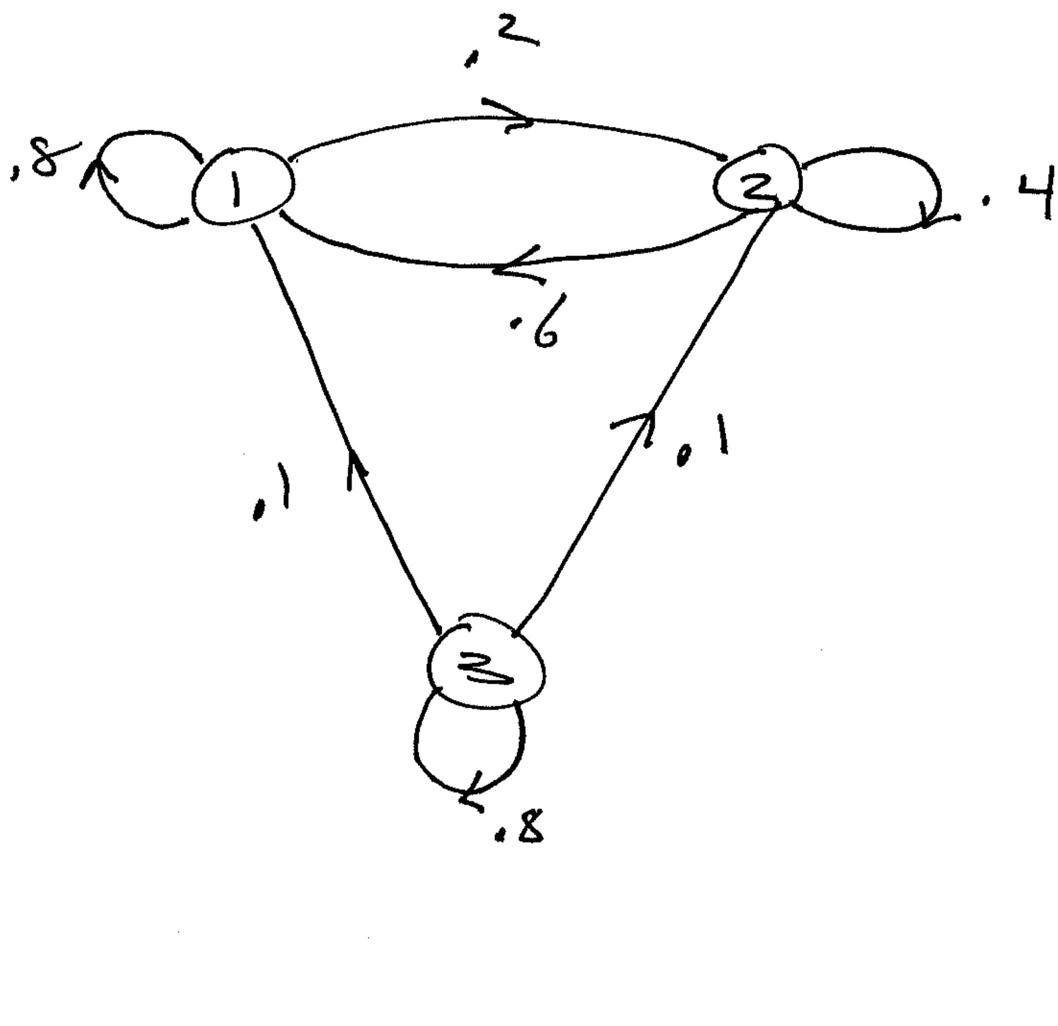
$0 < a < 1$

$0 < b < 1$

one can show

$$\lim_{n\to\infty} R^n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}$$

Ex.



Thus

$$R = \begin{pmatrix} .8 & .2 & 0 \\ .6 & .4 & 0 \\ .1 & .1 & .8 \end{pmatrix}$$

one can show

$$\lim_{n \to \infty} R^n = \begin{pmatrix} .75 & .25 & 0 \\ .75 & .25 & 0 \\ .75 & .25 & 0 \end{pmatrix}$$

# 7.2 Classification of States

Defn

We say $j$ is <u>accessible from</u> $i \in S$ (or that $j$ is <u>reachable</u> from $i$) iff

$$r_{ij}(n) > 0 \quad \text{for some } n \geq 1$$

Equivalently there exists a seq. of states $i, i_1, i_2, \ldots, i_{n-1}, j \in S$ such that transitions

$$i \to i_1 \to i_2 \to \cdots \to i_{n-1} \to j$$

all have positive probability.

notation: $(i) \dashrightarrow (j)$

Defn

let $A(i)$ denote the set of states accessible from $i$

$$A(i) = \{ j \in S \mid (i) \to (j) \}$$

Defn

A state $i$ is called recurrent iff for every $j$ accessible from $i$, also have $i$ accessible from $j$.

equivalently:

$$j \in A(i) \text{ iff } i \in A(j)$$

i.e.

$$(i) \dashrightarrow (j) \text{ iff } (j) \dashrightarrow (i)$$

Hence, a recurrent state is visited at all, will be visited infinitely often.

<u>Defn</u>

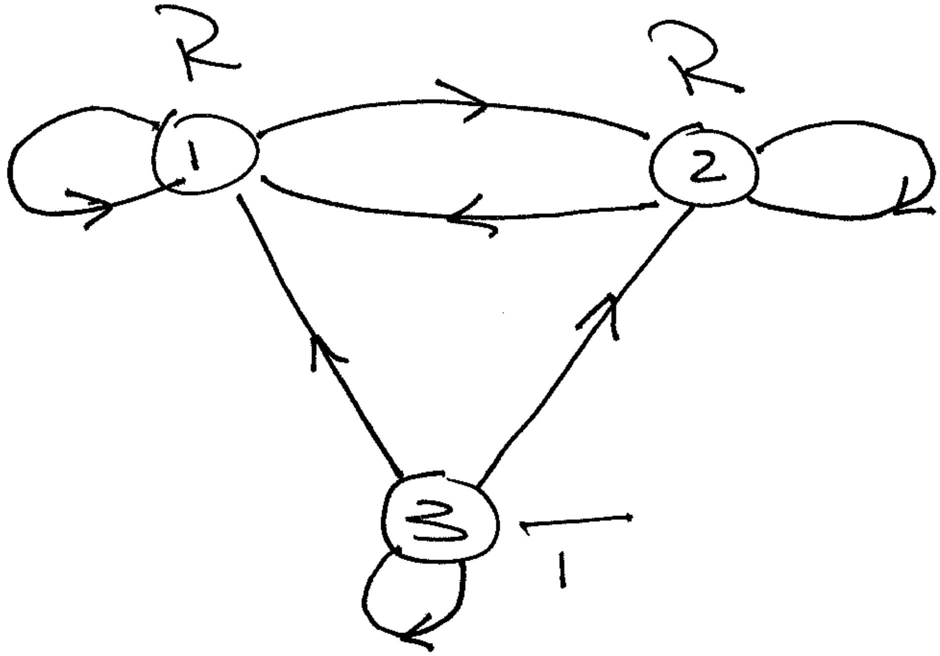
A state $i$ is called transient iff it is not recurrent. Thus $i$ is transient iff there exists a state $j$ with

$$j \in A(i) \text{ and } i \notin A(j)$$

Thus, after visiting a transient state i, there is a positive Prob. of entering such a state j. So, given enough time, this will happen.

Hence, a transient state will be visited only a finite # of times.

Ex.




Ex