CSE 107 5-8-25

3.4 Joint PDFs

Defn

We say r.v.s X, Y are jointly continuous iff there exists a function $f_{X,Y}(x,y)$ s.t.

$$P((X,Y) \in S) = \iint_S f_{X,Y}(x,y)\,dx\,dy$$

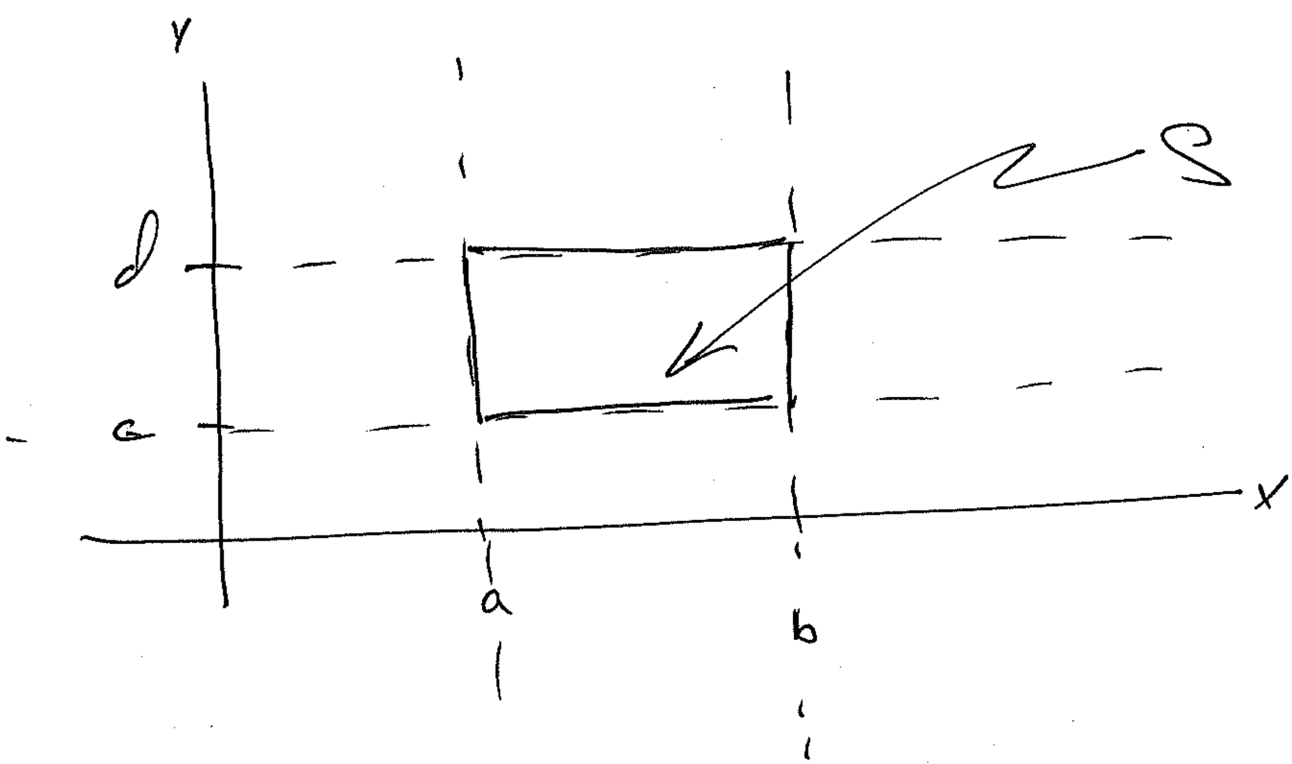
for 'any' subset $S \subseteq \mathbb{R}^2$. We call $f_{X,Y}(x,y)$ the joint PDF of X, Y.

In Particular, if

$$S = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$$
$$= [a,b] \times [c,d]$$

then

$$P(a \le X \le b,\ c \le Y \le d) = \int_c^d \int_a^b f_{X,Y}(x,y)\,dx\,dy$$



If $S = \mathbb{R}^2$, we have

$$1 = P(\Omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

If $\delta > 0$ is small, and $f_{X,Y}(x,y)$ is continuous near (a,c) then

$$P(a \le X \le a+\delta,\ c \le Y \le c+\delta) \leftarrow \text{'mass'}$$

$$= \int_a^{a+\delta}\int_c^{c+\delta} f_{X,Y}(x,y)\,dy\,dx \approx \underbrace{f_{X,Y}(a,c)}_{\substack{\text{mass/area} \\ \text{density}}} \cdot \underbrace{\delta^2}_{\text{area}}$$

Recall, for $T \subseteq \mathbb{R}$, we have

(1) $$P(X \in T) = \int_T f_X(x)\,dx$$

also

$$P(X \in T) = P(X \in T, \Omega)$$

$$= P(X \in T,\ Y \in \mathbb{R})$$

$$= \iint_{T \times \mathbb{R}} f_{X,Y}(x,y)\,dx\,dy$$

(2) $$= \int_T \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \right) dx$$

comparing (1) & (2), we get

$$\boxed{f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy}$$

also

$$\boxed{f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}$$

also

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \right) dx$$

$$= \iint_{\mathbb{R}^2} x\, f_{X,Y}(x,y)\,dx\,dy$$

Thus

$$\boxed{\begin{array}{l} E[X] = \displaystyle\iint_{\mathbb{R}^2} x\, f_{X,Y}(x,y)\,dx\,dy \\ \hline E[Y] = \displaystyle\iint_{\mathbb{R}^2} y\, f_{X,Y}(x,y)\,dx\,dy \end{array}}$$

Ex. Joint uniform continuous r.v.s on $R = [a,b] \times [c,d]$

$$f_{X,Y}(x,y) = \begin{cases} \alpha & (x,y) \in R \\ 0 & (x,y) \notin R \end{cases}$$

must have

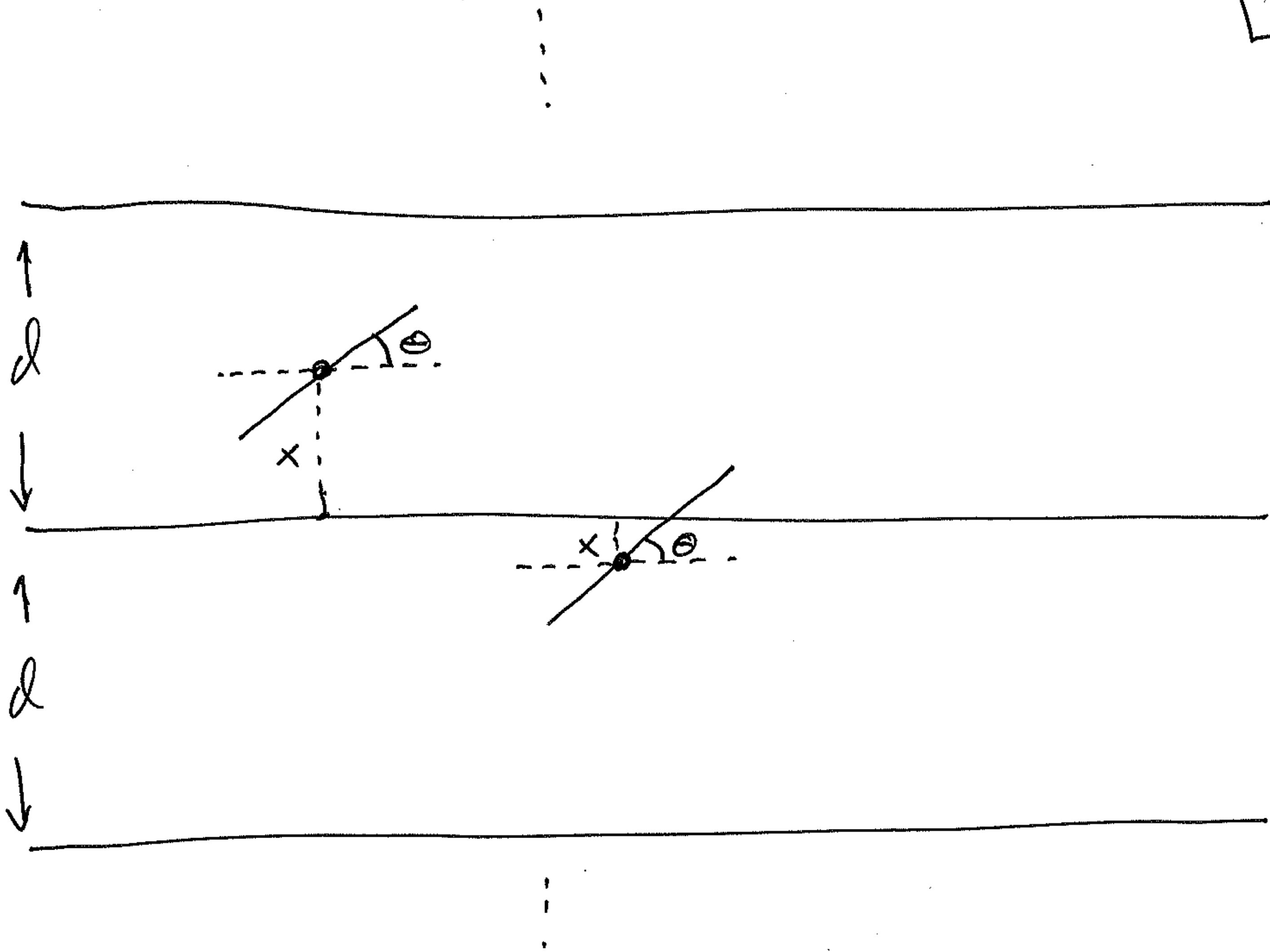
$$1 = \iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = \int_a^b \int_c^d \underbrace{f_{X,Y}(x,y)}_{\alpha}\,dy\,dx$$

$$\therefore \quad 1 = \alpha \cdot \text{area}(R)$$

$$\therefore \quad \alpha = \frac{1}{\text{area}(R)}$$

Ex. Buffon's Needle

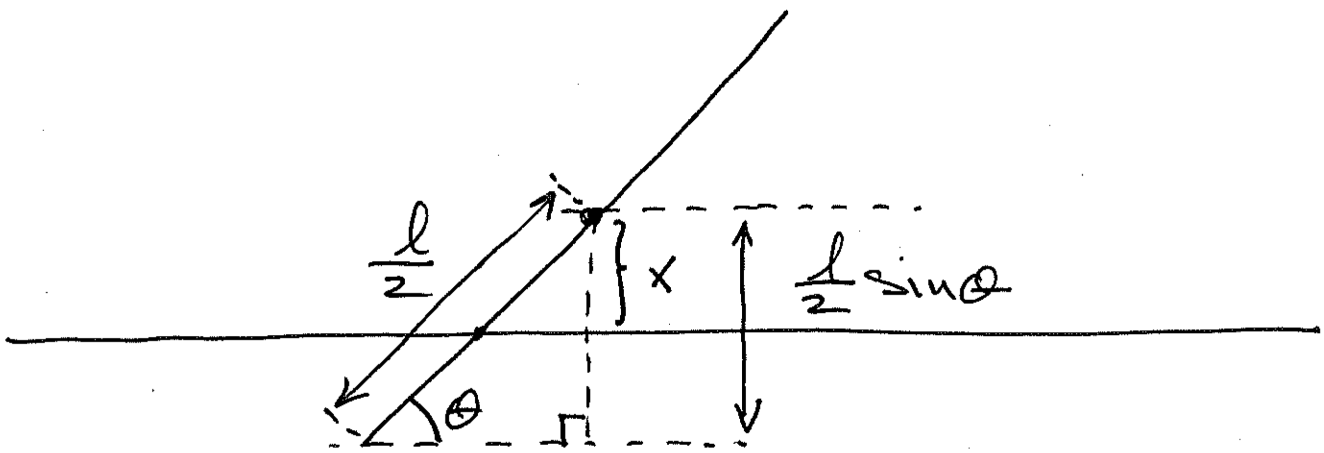
A surface is ruled with parallel lines, distance d apart. A needle of length l is tossed on the surface. What is the probability that the needle intersects one of the lines. (assume $l \le d$, so it can't intersect 2 lines.)



Let X be vertical dist. from midpoint of needle to nearest line, and Θ the acute angle between needle and horizontal.

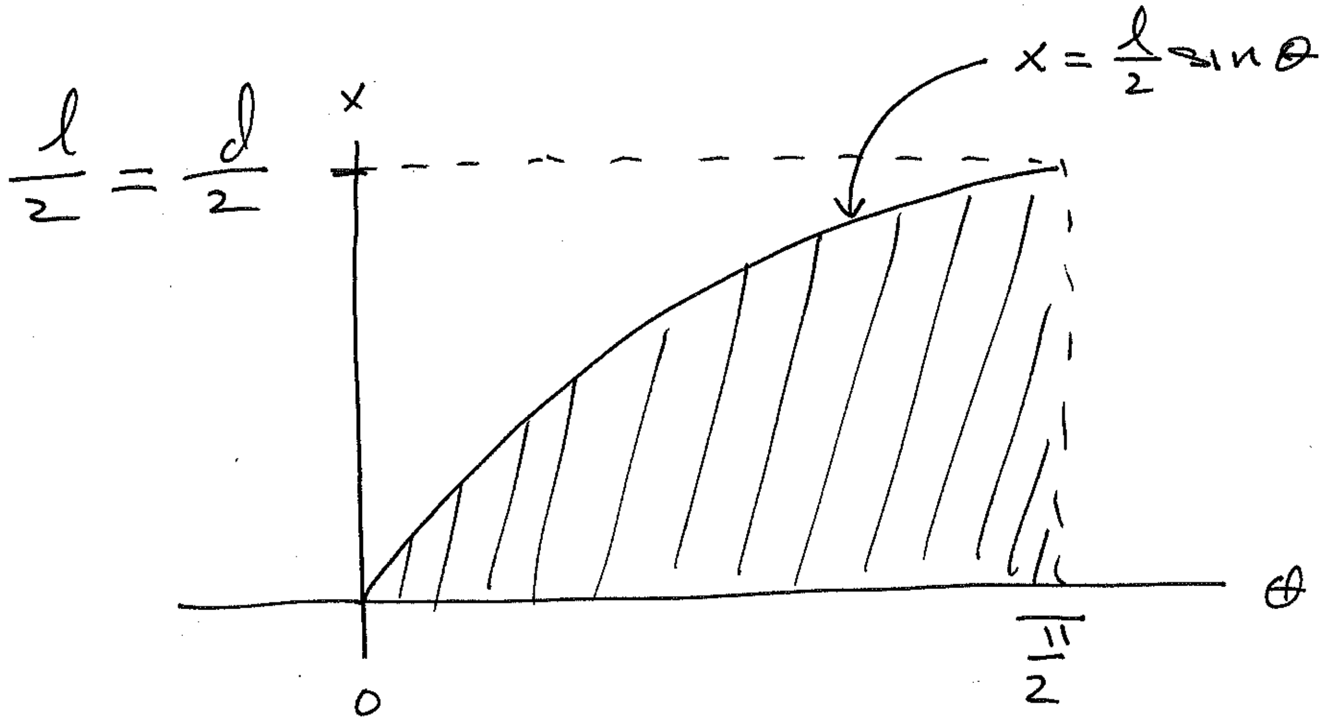
Model (X, Θ) with a joint uniform PDF on $[0, \frac{d}{2}] \times [0, \frac{\pi}{2}]$, with area $= \frac{\pi d}{4}$. Thus

$$f_{X,\Theta}(x,\theta) = \begin{cases} \frac{4}{\pi d} & 0 \le x \le \frac{d}{2},\ 0 \le \theta \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$



Thus intersection occurs iff

$$X \le \frac{l}{2}\sin\theta$$



Therefore

$$P\left(X \le \frac{l}{2}\sin\Theta\right) = \iint\limits_{x \le \frac{l}{2}\sin\theta} f_{X,\Theta}(x,\theta)\,dx\,d\theta$$

$$= \frac{4}{\pi d}\int_0^{\pi/2}\int_0^{\frac{l}{2}\sin\theta} dx\,d\theta$$

$$= \frac{4}{\pi d}\int_0^{\pi/2} \frac{l}{2}\sin\theta\,d\theta$$

$$= \frac{4}{\pi d} \cdot \frac{\ell}{2} \left(-\cos\theta\right)\Big|_0^{\pi/2}$$

$$= \frac{2\ell}{\pi d}\left((-0)-(-1)\right)$$

$$= \frac{2\ell}{\pi d}$$

Pick $\ell = d$, then

$$P(\text{intersection}) = \frac{2}{\pi}$$

$$\therefore \quad \pi = \frac{2}{P(\text{intersect})}$$

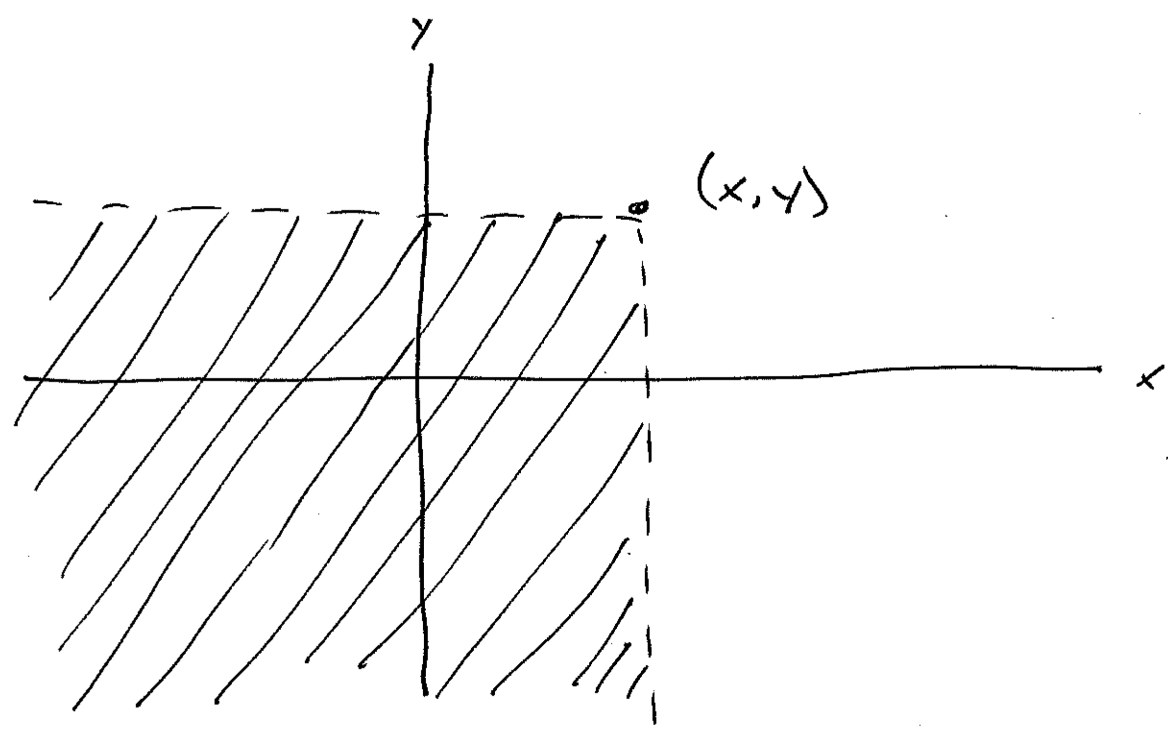
$$\therefore \quad \pi \approx \frac{2}{\left(\dfrac{\#\text{intersect}}{\#\text{tosses}}\right)} = \frac{2 \cdot \#\text{tosses}}{\#\text{intersections}}$$

Joint CDFs

If X, Y are two r.v.s, their Joint CDF

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\, dt\, ds$$



note

$$f_{X,Y}(x,y) = \frac{\partial^2 F}{\partial y \, \partial x}(x,y)$$

If $g : \mathbb{R}^2 \to \mathbb{R}$, then $g(X,Y)$ is a new r.v..

Expected value rule

$$E[g(X,Y)] = \iint_{\mathbb{R}^2} g(x,y) \cdot f_{X,Y}(x,y) \, dx \, dy$$

special case:

$$g(x,y) = ax + by + c$$

so

$$E[aX+bY+c]$$

$$= \iint_{\mathbb{R}^2} (ax+by+c) f_{X,Y}(x,y)\,dx\,dy$$

$$= a\int_{-\infty}^{\infty} x \cdot \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx$$

$$+ b\int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

$$+ c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

$$= aE[X] + bE[Y] + c$$

3.5 conditioning

As before, we condition on $A \subseteq \Omega$, with $P(A) > 0$.

let X be a cont. r.v.

Defn

The conditional PDF of X given A is a fcn. $f_{X|A}(x)$ such that

$$P(X \in I \mid A) = \int_I f_{X|A}(x)\, dx$$

for 'any' subset $I \subseteq \mathbb{R}$.

we require

$$\int_{-\infty}^{\infty} f_{X|A}(x)\,dx = 1$$

special case: $A = \{X \in J\}$ for some $J \in \mathbb{R}$, $P(A) > 0$.
Then for any $I \subseteq \mathbb{R}$

$$P(X \in I \mid X \in J) = \frac{P(X \in I, X \in J)}{P(X \in J)}$$

$$= \frac{P(X \in I \cap J)}{P(X \in J)}$$

$$= \frac{\int_{I\cap J} f_X(x)\,dx}{P(X\in J)}$$

$$= \int_{I\cap J} \frac{f_X(x)}{P(X\in J)}\,dx$$

compare to defn of cond. PDF we have

$$f_{X|X\in J}(x) = \begin{cases} \dfrac{f_X(x)}{P(X\in J)} & \text{if } x\in J \\ 0 & \text{otherwise} \end{cases}$$

Ex.

The time T until a new light bulb turns out is exponential with parameter λ.

Alice turns on the light, leaves, then returns t_0 hours later, finding light still on.

Thus the event

$$A = \{T > t_0\}$$

has occurred. Let X be the additional time to burn out.

$$\therefore \quad X = T - t_0 \quad , \quad T = X + t_0$$

Find the cond. PDF

$$f_{X|A}(x)$$

<u>Soln</u>

1st find conditional CDF

$$F_{X|A}(x) = P(X \le x \mid A)$$

$$= \int_{-\infty}^{x} f_{X|A}(s)\, ds$$

Then differentiate to get

$$f_{X|A}(x) = \frac{d}{dx} F_{X|A}(x)$$

Recall

$$P(T > a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \\ 1 & \text{if } a \leq 0 \end{cases}$$

Thus for $x \geq 0$:

$$P(X > x \mid A) = P(T > t_0 + x \mid T > t_0)$$

$$= \frac{P(T > t_0 + x, T > t_0)}{P(T > t_0)}$$

$$= \frac{P(T > t_0 + x)}{P(T > t_0)} \quad \left\{ \begin{array}{l} \text{since} \\ \{T > t_0 + x\} \\ \subseteq \{T > t_0\} \end{array} \right.$$

$$= \frac{e^{-\lambda(t_0 + x)}}{e^{-\lambda t_0}}$$

$$= \frac{\cancel{e^{-\lambda t_0}} \cdot e^{-\lambda x}}{\cancel{e^{-\lambda t_0}}} = e^{-\lambda x}$$

$$\therefore F_{X|A}(x) = P(X \le x \mid A)$$

$$= 1 - P(X > x \mid A)$$

$$= \begin{cases} 1 - e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$\therefore f_{X|A}(x) = \underbrace{\begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}}_{\text{exponential}(\lambda)}$$