CSE 107 5-6-25

Recall: if X is Geometric(p)

$$p_X(k) = p(1-p)^{k-1} \quad (k=1,2,3,\dots)$$

Thus

$$F_{geo}(n) = \sum_{k=1}^{n} p(1-p)^{k-1}$$

$$= p \cdot \sum_{k=0}^{n-1} (1-p)^k$$

$$= \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{1-(1-p)}}$$

$$= 1-(1-p)^n$$

If Y is Exponential(λ) then

$$f_Y(x) = \lambda e^{-\lambda x} \quad (x \geq 0)$$

Thus for $x > 0$:

$$F_{exp}(x) = \int_0^x \lambda e^{-\lambda t}\, dt$$

$$= -e^{-\lambda t}\Big|_0^x = -\left(e^{-\lambda x} - 1\right)$$

$$= 1 - e^{-\lambda x}$$

To compare F_{geo}, F_{exp} pick $\delta > 0$ so that

$$\boxed{e^{-\lambda \delta} = 1 - p}$$

i.e. $p = 1 - e^{-\lambda\delta}$

$\therefore \quad -\lambda\delta = \ln(1-p)$

$\therefore \quad \delta = \dfrac{-\ln(1-p)}{\lambda}$

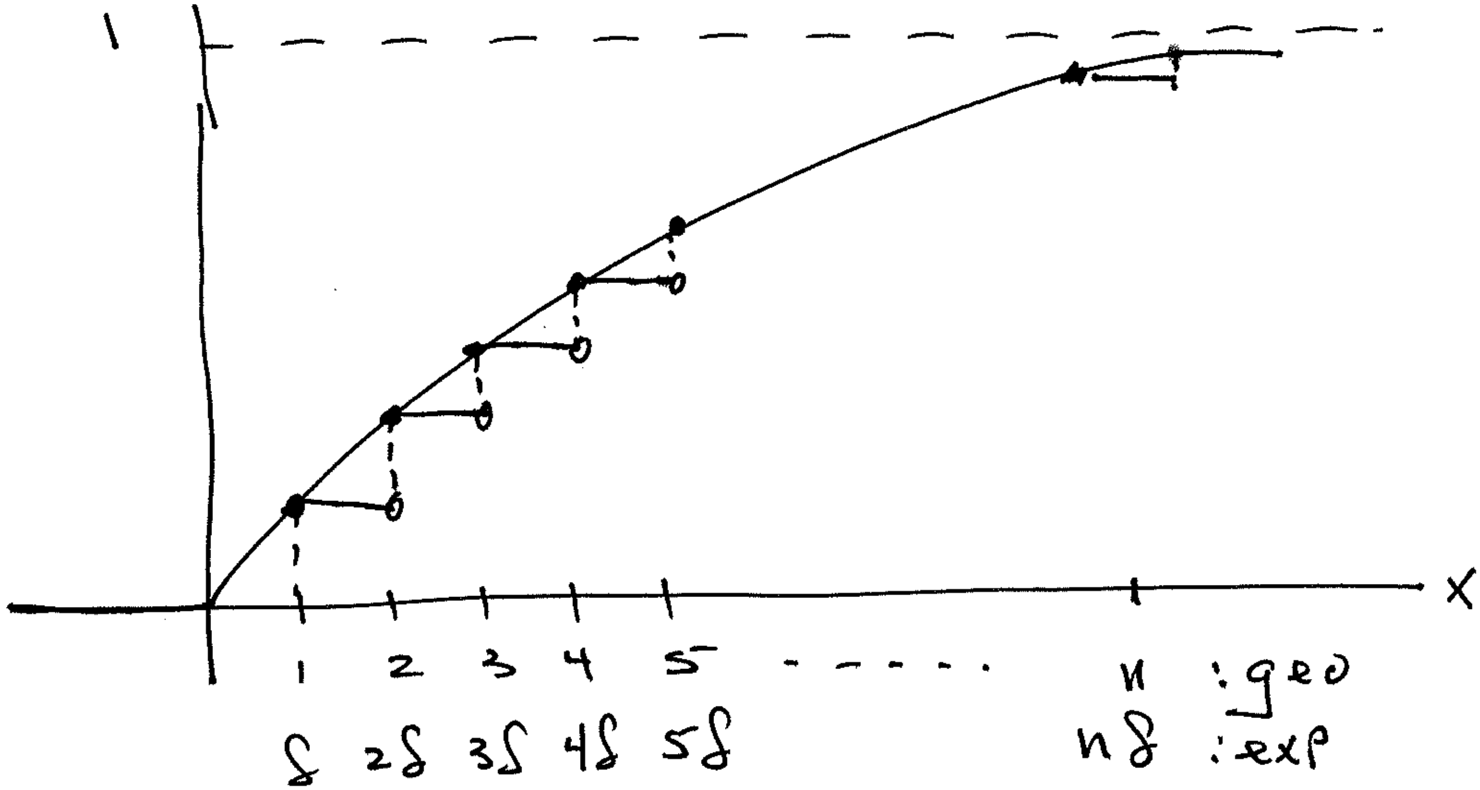
For this δ, we have

$$\left(e^{-\lambda\delta}\right)^n = (1-p)^n$$

$\therefore \quad 1 - e^{-\lambda\delta n} = 1 - (1-p)^n$

$\therefore \quad \overline{F}_{exp}(\delta n) = \overline{F}_{geo}(n)$

Graph



If we toss a coin quickly with

$$p = 1 - e^{-\lambda\delta}$$

We get $F_{geo} \longrightarrow F_{exp}$

3.3 Normal random Variable

Let X be a continuous r.v. with PDF

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}} \quad (x \in \mathbb{R})$$

X is called <u>Normal</u> or <u>Gaussian</u>.

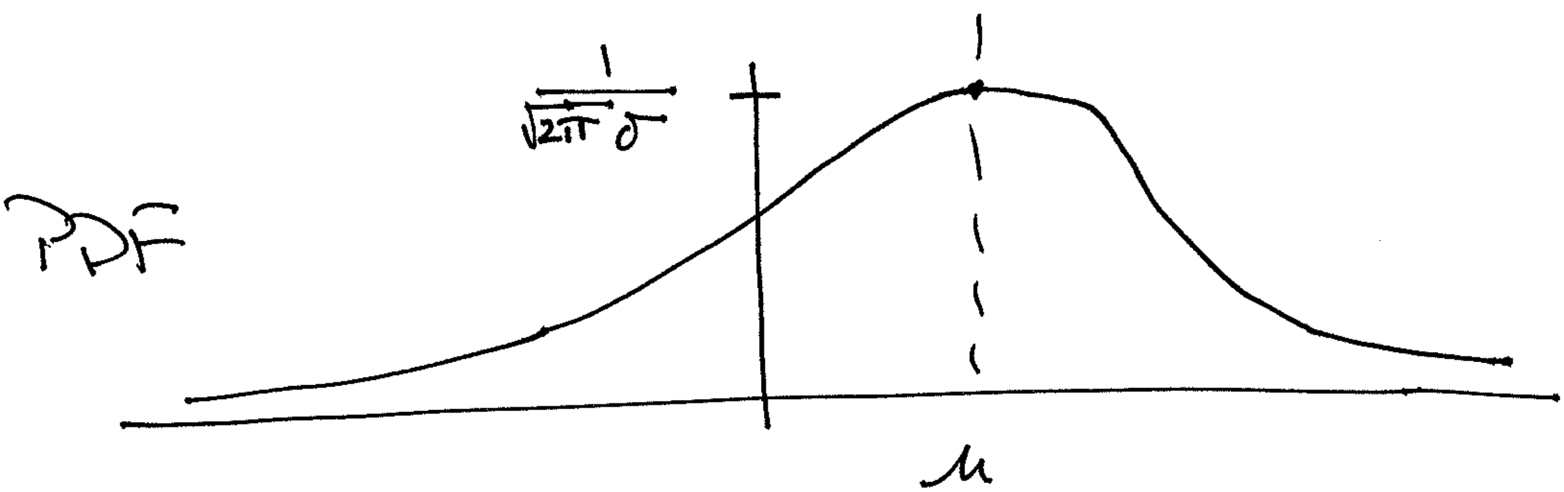
<u>Fact</u>:

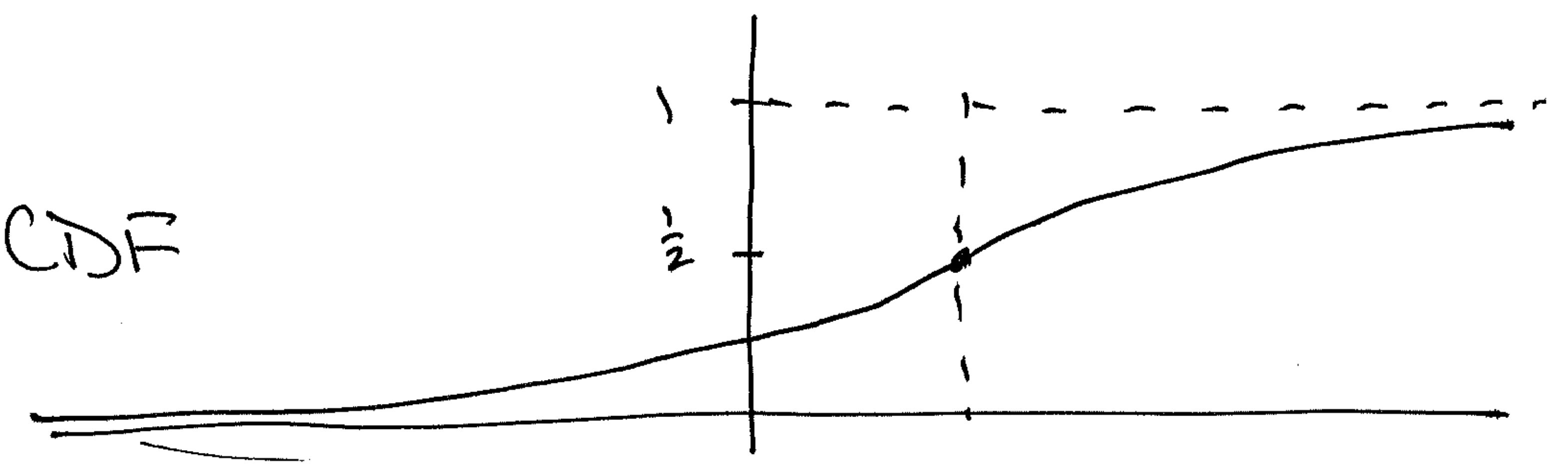
$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

See Handout: <u>Gaussian integral</u>.

Facts:

- $E[X] = \mu$
- $Var(X) = \sigma^2$
- $\sqrt{Var(X)} = \sigma$





Thm

for any PDF f_X, if

$$f_X(x) = f_X(-x) \quad \forall x \in \mathbb{R}$$

then

$$E[X] = 0$$

Proof

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$= \underbrace{\int_{-\infty}^{0} x f_X(x) dx}_{} + \int_{0}^{\infty} x f_X(x) dx$$

$t = -x \quad dt = -dx$

$x = -t \quad dx = -dt$

$x \leq 0 \leftrightarrow t \geq 0$

$$= \int_{\infty}^{0} \cancel{-}t\, f_X\!\left(\overset{\text{symm}}{\cancel{-}t}\right)(\cancel{-1})\,dt + \int_0^{\infty} x\, f_X(x)\,dx$$

$$= -\int_0^{\infty} t\, f_X(t)\,dt + \int_0^{\infty} x\, f_X(x)\,dx$$

$$= 0 \qquad \blacksquare$$

<u>Theorem</u>

If X is normal (μ, σ^2), and if $a \neq 0$, then

$$Y = aX + b$$

is also <u>normal</u> with $E[Y] = a\mu + b$
$\mathrm{Var}(Y) = a^2\sigma^2$

Proof

let $Y = aX + b$ with $a > 0$.

$$F_Y(y) = P(Y \le y)$$
$$= P(aX + b \le y)$$
$$= P\left(X \le \frac{y-b}{a}\right)$$

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right)$$

$\frac{d}{dy} \downarrow$

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$
$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{\left(\frac{y-b}{a} - \mu\right)^2}{2\sigma^2}}$$
$$= \frac{1}{\sqrt{2\pi} \cdot (a\sigma)} \cdot e^{-\frac{\left(\frac{y-b-a\mu}{a}\right)^2}{2\sigma^2}}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}\cdot(a\sigma)} \cdot e^{-\frac{(y-(a\mu+b))^2}{2(a\sigma)^2}}}$$

normal with mean $a\mu+b$ and s.d. $a\sigma$, so Var $a^2\sigma^2$

■

Exercise do case $a<0$.

A normal r.v. Y is called standard normal iff its mean is $\boxed{0}$, and its variance is $\boxed{1}$

PDF : $\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}$

The CDF of the std. normal r.v. is denoted

$$\Phi(y) = P(Y \le y)$$

$$= P(Y < y)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} \, dt$$

Handout: <u>Standard normal table</u>

gives values of $\Phi(y)$ for

$$0 \le y \le 3.49$$

If $y > 3.49$, $\Phi(y) \approx 1$. The identity

$$\Phi(-y) = 1 - \Phi(y) \qquad (y \in \mathbb{R})$$

gives us $\Phi(y)$ for $y < 0$.

<u>Ex.</u>

- $\Phi(1.84) = \boxed{.9671}$
- $\Phi(-1.84) = 1 - \Phi(1.84)$
  $= 1 - (.9671) = \boxed{.0329}$

If X is normal with mean μ, s.d. σ, Let

$$Y = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}.$$

Then Y is normal with

$$E[Y] = \frac{1}{\sigma}E[X] - \frac{\mu}{\sigma} = \frac{\mu}{\sigma} - \frac{\mu}{\sigma} = 0$$

and

$$Var(Y) = \frac{1}{\sigma^2}Var(X) = \frac{\sigma^2}{\sigma^2} = 1$$

$\therefore$ Y is standard normal.

Thus

$$F_X(x) = P(X \le x)$$
$$= P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$$
$$= P\left(Y \le \frac{x-\mu}{\sigma}\right)$$
$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

<u>Ex.</u> The mean high temp. in Santa Cruz on Nov. 1 is 66° F with s.d of 4.4° what is the probability that the high temp. will be $\le 57^\circ$. (assume normal.)

<u>Soln</u>

Let X be the high temp on Nov. 1. Then

$$P(X \le 57) = P\left(\frac{X-66}{4.4} \le \frac{57-66}{4.4}\right)$$

$$= P(Y \le -2.04)$$

$$= \Phi(-2.04)$$

$$= 1 - \Phi(2.04)$$

$$= 1 - (.9793)$$

$$\doteq \boxed{.0207}$$

Ex.

what is the prob. that a normal r.v. is within 1 s.d. of its mean?

soln

$$P(\mu-\sigma \le X \le \mu+\sigma)$$

$$= P\left(\frac{(\mu-\sigma)-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{(\mu+\sigma)-\mu}{\sigma}\right)$$

$$= P(-1 \le Y \le 1)$$

$$= P(Y \le 1) - P(Y < -1)$$

$$= \Phi(1) - \Phi(-1)$$

$= \Phi(1) - (1 - \Phi(1))$

$= \boxed{2\Phi(1) - 1}$

$= 2(.8413) - 1 = \boxed{.6826}$

Same Problem, but 2, 3 s.d.s!

- $P(X \text{ is within } 2\sigma \text{ of } \mu)$

$= 2\Phi(2) - 1 = 2(.9772) - 1 = \boxed{.9544}$

- $P(X \text{ is within } 3\sigma \text{ of } \mu)$

$= 2\Phi(3) - 1 = 2(.9987) - 1 = \boxed{.9974}$