CSE 107 5-30-25

Supplemental Lecture

Sum of Poisson r.v.s

Let X, Y be independent Poisson r.v.s with parameters λ_1, λ_2 resp.

Claim: $Z = X + Y$ is Poisson$(\lambda_1 + \lambda_2)$

Proof

we have

$$P_X(k) = e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \quad (k = 0, 1, 2, \ldots)$$

$$P_Y(k) = e^{-\lambda_2} \cdot \frac{\lambda_2^k}{k!} \quad (k = 0, 1, 2, \ldots)$$

Thus

$$P_Z(n) = \sum_{k=0}^{\infty} P_X(k) \cdot P_Y(n-k)$$

$$= \sum_{k=0}^{n} e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{(n-k)}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \lambda_1^k \cdot \lambda_2^{(n-k)}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \cdot \lambda_2^{(n-k)}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^n}{n!} \quad (n = 0, 1, 2, \dots)$$

$\therefore Z \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

$\blacksquare$

Ex

Rush hour is 8–9 am, during which accidents occur at rate 5/hr, as a Poisson Process. During period 9–11 am accidents occur, as a Poisson Process, with rate 3/hour.

What is the PMF of the total # of accidents from 8–11 am?

Soln

Let N_1 = # accidents in 8–9

N_2 = # " " 9–11

and $N = N_1 + N_2$ = # accidents in 8–11.

we know

$$N_1 \text{ is Poisson}(5 \cdot 1)$$

$$N_2 \text{ is Poisson}(3 \cdot 2 = 6)$$

Thus

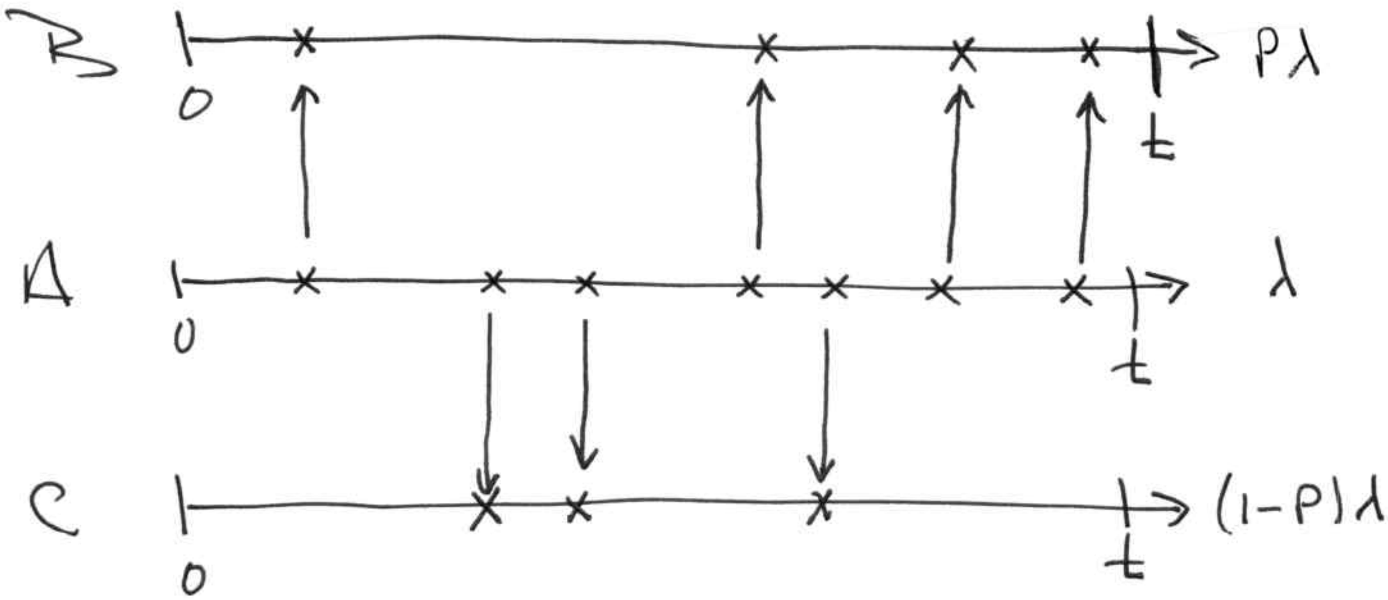
$$N = N_1 + N_2 \text{ is Poisson}(11)$$

and

$$\therefore \quad P_N(n) = e^{-11} \cdot \frac{11^n}{n!} \quad (n = 0, 1, 2, \ldots)$$

Splitting a Poisson Process

Let A be a Poisson Process of rate λ. Split A into two arrival processes B, C with probabilities p, $1-p$ respectively.



What kind of processes are B and C? Let

N = # arrivals of A in $[0,t]$

X = " " " B " "

Y = " " " C " "

We know N is Poisson(λt), i.e.

$$P_N(n) = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} \quad (n = 0, 1, \ldots)$$

Then

$$P_X(k) = P(X=k)$$

$$= \sum_{n=k}^{\infty} P(X=k \mid N=n) \cdot P(N=n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} \cdot p^k (1-p)^{n-k} \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{\cancel{n!}}{k!(n-k)!} p^k (1-p)^{n-k} e^{-\lambda t} \cdot \frac{(\lambda t)^n}{\cancel{n!}}$$

$$= \frac{p^k}{k!} e^{-\lambda t} \cdot \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} \cdot (\lambda t)^n}{(n-k)!} \quad \left\{ \begin{array}{l} \text{Sub} \\ n \leftarrow n+k \end{array} \right.$$

$$= \frac{p^k}{k!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{(1-p)^n \cdot (\lambda t)^{n+k}}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{((1-p)\lambda t)^n}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\lambda t} \cdot e^{(1-p)\lambda t}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\cancel{\lambda t} + \cancel{\lambda t} - p\lambda t}$$

$$= e^{-p\lambda t} \cdot \frac{(p\lambda t)^k}{k!}$$

Thus X is Poisson$(p\lambda t)$.

We see B satisfies: time-homogeniety, independence, and small interval probabilities.

$\therefore$ B is a Poisson Process of rate: $p\lambda$

Similarly, C is a Poisson Process of rate: $(1-p)\lambda$.

see example 6.13 on P. 318

Merging Poisson Processes

Ex.

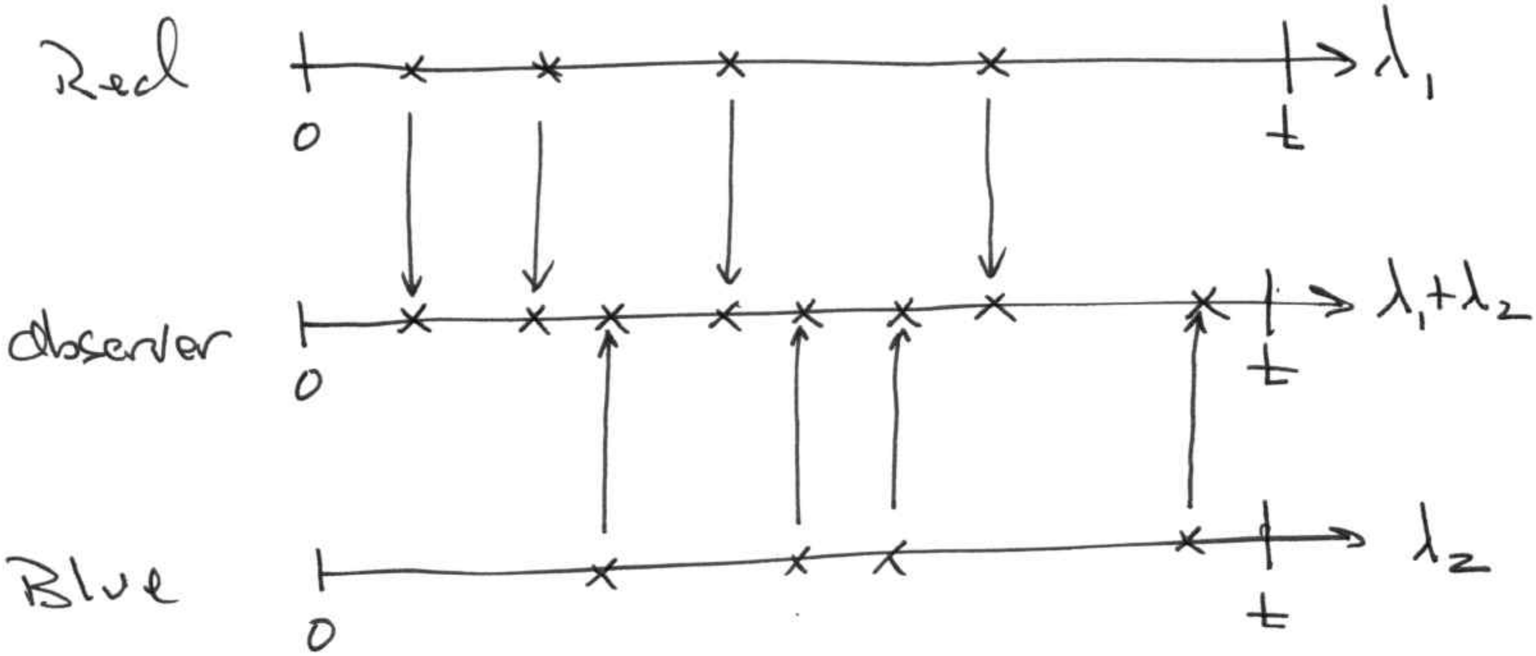
an observer sees 2 lights flashing at random times, Red & Blue.

Red: Poisson Process rate λ_1

Blue: " " " λ_2

The observer is colorblind however, so sees only flashing lights.

what kind of Process is the observer seeing. ?



Let

X = # Red lights in $[0, t]$

Y = " Blue " " "

M = # lights " "

Know: X is Poisson $(\lambda_1 t)$

Y is Poisson $(\lambda_2 t)$

Therefore $M = X + Y$ is itself Poisson$(\underbrace{\lambda_1 t + \lambda_2 t}_{(\lambda_1+\lambda_2)t})$. We see that the observer is seeing a Poisson Process at rate: $\lambda_1 + \lambda_2$.

Ex.

Suppose observer sees a single flash. What is the prob. that it was Red?

Soln (use small interval Probabilities)

Let $\delta > 0$ be a small time interval containing the flash. We seek

$P(1 \text{ Red} \mid 1 \text{ light})$

$$= \frac{P(1 \text{ Red}, 1 \text{ light})}{P(1 \text{ light})}$$

$$= \frac{P(1 \text{ Red})}{P(1 \text{ light})}$$

$$= \frac{\lambda_1 \delta}{(\lambda_1+\lambda_2)\delta} = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

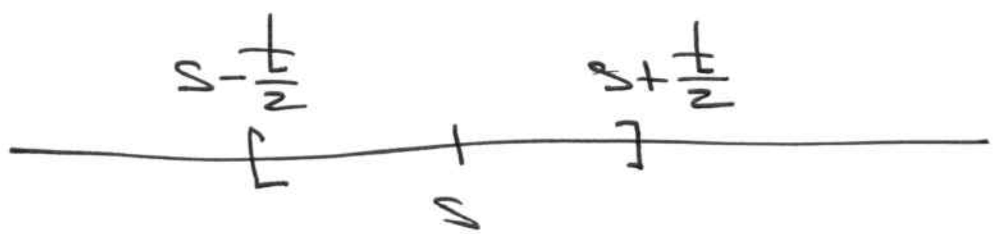
Similarly, Probability that observer saw a Blue light is

$$\frac{\lambda_2}{\lambda_1+\lambda_2}.$$

Can we do this without small interval probabilities?

Solution 2

Suppose light flashes at time s.
Consider a time interval centered at s, with length t



i.e. $I_t = \left[s-\frac{t}{2}, s+\frac{t}{2}\right]$. Then

$$P(1 \text{ Red in } I_t \mid 1 \text{ light in } I_t)$$

$$= \frac{P(1 \text{ Red in } I_2, 1 \text{ light in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{P(1 \text{ Red in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{e^{-\lambda_1 t} \cdot \frac{(\lambda_1 t)^1}{1!}}{e^{-(\lambda_1+\lambda_2)t} \cdot \frac{((\lambda_1+\lambda_2)t)^1}{1!}}$$

$$= e^{\lambda_2 t} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} \xrightarrow[t \to 0]{\text{as}} \frac{\lambda_1}{\lambda_1+\lambda_2}$$