CSE 107 5-29-25

Exercises (6.1)

- first split, then Merge.
  determine the resulting Param.
- first merge, then split.
  determine Parameters.
- change bit operation in merge
  from <u>or</u> to <u>and</u>, <u>exor</u>, <u>nand</u>, <u>nor</u>.
  determine Parameters.

6.2 Poisson Process

Recall: a Bernoulli Process

$$\{X_t \mid t = 1, 2, 3, \dots\}$$

satisfies

(1) each X_t is Bernoulli(p)

(2) the set $\{X_t \mid t = 1, 2, \dots\}$ is independent, equivalently, events in disjoint time intervals are independent.

we defined

Y_k = time to k^{th} arrival

- Y_k is Pascal(p, k)

- $P_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k} \quad t = k, k+1, \ldots$

we defined

$T_1 = Y_1$

$T_k = Y_k - Y_{k-1}$

- T_k is geometric(p)

- $P_{T_k}(t) = (1-p)^{t-1} \cdot p, \quad t = 1, 2, 3, \ldots$

can also define

$$N_t = X_1 + X_2 + \cdots + X_t$$

$$= \# \text{ arrivals in } [1, t]$$

- N_t is Binomial(t, p)
- $P_{N_t}(k) = \binom{t}{k} p^k (1-p)^{t-k}, \quad k = 0, 1, \ldots, t$

We can recover X_t from N_t by

$$X_1 = N_1$$

$$X_t = N_t - N_{t-1} \qquad t = 2, 3, \ldots$$

also

$$Y_k = \min\{t \mid N_t = k\} \qquad k = 1, 2, \ldots$$

Thus we can use any of the four sequences X_t, N_t, Y_k, T_k to define a Bernoulli Process.

A <u>Poisson Process</u> the continuous time analog of Bernoulli Process.

Consider an arrival Process that evolves in continuous time.

Define

$$P(k, \tau) = P(\#\text{ of arrivals during an interval of len. } \tau)$$

<u>Defn</u>

An arrival Process is a <u>Poisson Process</u> iff

(a) <u>time homogeneity</u>

$P(k, \tau)$ is the same for all intervals of length τ : $[t, t+\tau]$

(b) <u>independence</u>

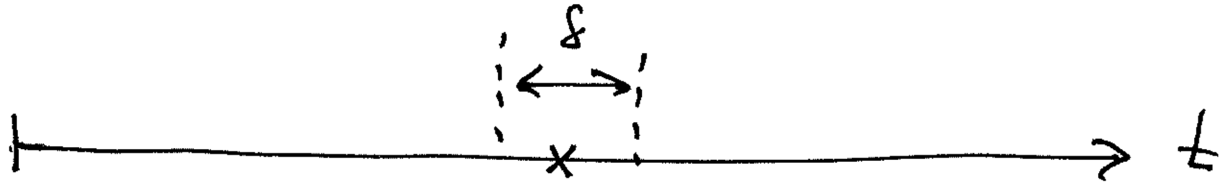
The #s of arrivals in disjoint time intervals are independent.

(c) <u>Small interval Probabilities</u>

let $\delta > 0$ be small. Then

$$P(k, \delta) \approx \begin{cases} 1 - \lambda\delta & \text{if } k = 0 \\ \lambda\delta & \text{if } k = 1 \\ 0 & \text{if } k = 2, 3, \ldots \end{cases}$$

$\lambda > 0$ is the '<u>arrival rate</u>'



The approx. $\approx$ refers to the limit $\delta \to 0$

$$P(k,\delta) = \begin{Bmatrix} 1-\lambda\delta & k=0 \\ \lambda\delta & k=1 \\ 0 & k\geq 2 \end{Bmatrix} + o(\delta)$$

where $o(\delta)$ denotes a function with the property

$$\lim_{\delta \to 0} \left(\frac{o(\delta)}{\delta} \right) = 0$$

for instance the function $c \cdot \delta^2$ has this property.

observe

$$E[\#\text{arrivals in } [t, t+\delta]]$$

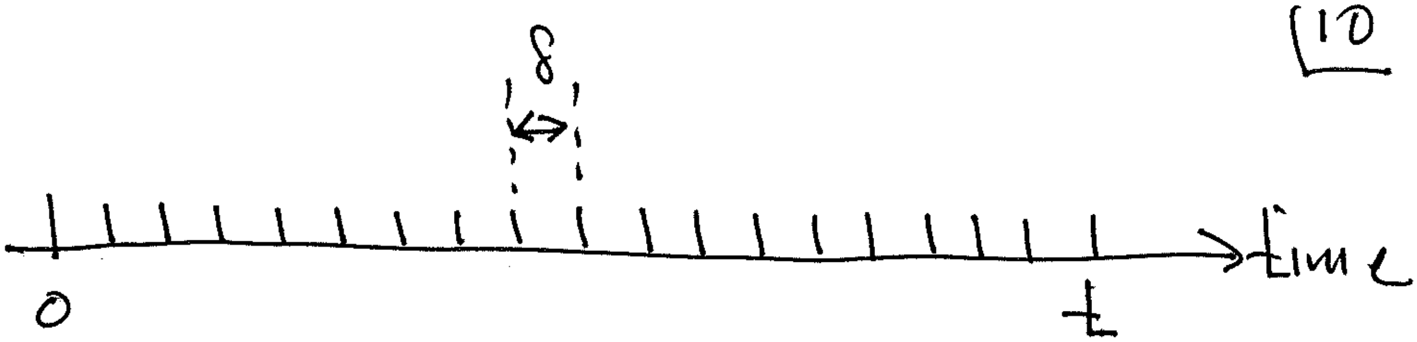
$$\approx (1-\lambda\delta)\cdot 0 + \lambda\delta \cdot 1 = \lambda\delta$$

#arrivals

time

$\therefore$ λ is #arrivals per unit time.

To reason about large time intervals, we divide into small intervals.



Let $n = \frac{t}{\delta}$ = # of sub-intervals.

choose $\delta > 0$ so small that Prob. of more than one arrival in a sub-interval is negligable.

We have (approximately) a Bernoulli Process, with Prob. of arrival

$$p = \lambda \delta = \frac{\lambda t}{n}$$

approx. improves as $\delta \to 0$ $(n \to \infty)$.

The # arrivals in Bernoulli Process is Binomial(n, p).

so approximately

$$P(k \text{ arrivals in } [0, t])$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k}$$

In the limit as $n \to \infty$ $(\delta \to 0)$ this becomes Poisson(λt).

(from handout.)

So

$$P(k \text{ arrivals in } [0,t]) = e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}$$

Let N_t = # arrivals in $[0,t]$ for a Poisson Process.

Then

$$P_{N_t}(k) = \begin{cases} e^{-\lambda t} \cdot \dfrac{(\lambda t)^k}{k!} & k = 0, 1, 2 \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$E[N_t] = \lambda t$$

and

$$Var(N_t) = \lambda t$$

Let T be the time to 1^{st} arrival in a Poisson process at rate λ.

<u>Goal</u>: derive dist. of T.

observe

$$\{T > t\} = \{N_t = 0\}$$

Thus

$$F_T(t) = P(T \le t)$$

$$= 1 - P(T > t)$$

$$= 1 - P(N_t = 0)$$

$$= 1 - P_{N_t}(0)$$

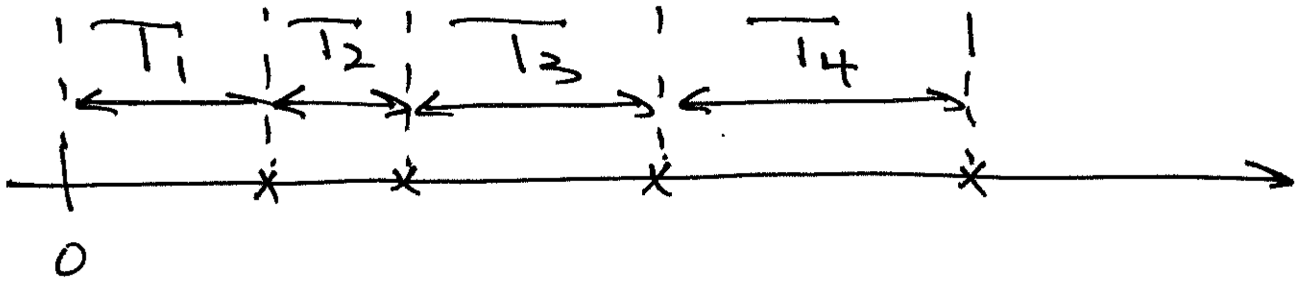
$$= 1 - e^{-\lambda t}$$

diff w.r.t. t to find

$$\boxed{f_T(t) = \lambda e^{-\lambda t} \quad \text{for } t \ge 0}$$

$\therefore$ T is exponential(λ)

Let T_K be K^{th} interarrival time



After each arrival Poisson Process has a 'fresh start',

so

$$f_{T_K}(t) = \lambda e^{-\lambda t} \quad \text{for } t \geq 0$$

Exponential (λ)

Now define

$$Y_K = T_1 + T_2 + \cdots + T_K \quad (K = 1, 2, \ldots)$$

$$= \text{time to } K^{th} \text{ arrival}$$

Y_K has Erlang distribution of order K

$$f_{Y_K}(t) = \frac{\lambda^K t^{K-1} e^{-\lambda t}}{(K-1)!} \quad (t \geq 0$$

(see derivation in text.)

Exercise

Prove that the sum of K independent exponential(λ) r.v.s is Erlang of order K.

Hint

use induction on K.

- Base: $Y_1 = T_1$
- Induction:

$$\underbrace{Y_K}_{\substack{\text{Erlang}\\\text{order } K}} = \underbrace{\underbrace{Y_{K-1}}_{\substack{\text{ind.}\\\text{hyp}}} + \underbrace{T_K}_{\text{exp}}}_{\text{convolution}}$$

Summary

	Bernoulli	Poisson
time:	$t \in \{1, 2, \ldots\}$	$t \in [0, \infty)$
rate:	p/trial	λ/unit-time
N_t:	Binomial(t, p)	Poisson(λt)
T_k:	Geometric(p)	Exponential(λ)
Y_k:	Pascal(p, k)	Erlang(λ, k)

Ex.

You receive email in a Poisson Process with rate 2/hour currently have no messages.

a) what is the Prob. of having 3 new msgs. in 30 minutes

$$P(N_{\frac{1}{2}}=3)=P_{N_{\frac{1}{2}}}(3)$$

$$= e^{-(2\cdot\frac{1}{2})}\cdot\frac{(2\cdot\frac{1}{2})^3}{3!}$$

$$= \frac{1}{6e} \approx \boxed{.0613}$$

b.) What's Prob. of having no msgs in 30 min?

$$P(N_{\frac{1}{2}}=0)=P_{N_{\frac{1}{2}}}(0)$$

$$= e^{-(2\cdot\frac{1}{2})}\cdot\frac{(2\cdot\frac{1}{2})^0}{0!}$$

$$= e^{-1} = \boxed{.3679}$$

c.) What is expected time of 5th new msg?

$$E[Y_5] = E[T_1 + \cdots + T_5]$$

$$= 5\cdot\frac{1}{2} = \frac{5}{2}$$

$$= \boxed{2.5 \text{ hours}}.$$