CSE 107 5-27-25

6.1 The Bernoulli Process

Defn

The Bernoulli Process is a sequence

$$X_1, X_2, \ldots\ldots$$

of independent Bernoulli r.v.s with

$$P(X_t = 1) = p$$

$$P(X_t = 0) = 1-p$$

for $t = 1, 2, 3, \ldots\ldots$

Two ways to think of this

- one experiment, repeated ∞-many times: $\Omega = \{0, 1\}$

- a single experiment with $\Omega = \{\text{infinite bit strings}\}$

In 2^{nd} view, suppose $0 < p < 1$ and let

$\omega = 111 \cdots$ (all 1's)

Then

$$P(\omega) = P(X_t = 1 : \forall t)$$
$$\leq P(X_t = 1 : 1 \leq t \leq n) = p^n$$

Since $p^n \to 0$ as $n \to \infty$

$P(\omega) = 0$

similarly $P(\omega) = 0$ for any $\omega \in \Omega$.

Consider 1st view: we have basically 2 questions.

(1) given a number of trials, how many successes occurred?

Binomial(n, p)

② how many trials until first success?

Geometric(p)

we know if T is time to 1st success

$$p_T(t) = (1-p)^{t-1} p \quad (t = 1, 2, \ldots)$$

$$E[T] = \frac{1}{p}$$

$$Var(T) = \frac{1-p}{p^2}$$

Recall <u>Memorylessness</u>:

$$P(T-n=t \mid T>n) =$$

$$= \cdots\cdots$$

$$= P(T=t)$$

(also called <u>fresh start</u>)

<u>Inter-arrival times</u>

<u>note</u> 'success' = 'arrival'

let

T_1 = time to 1^{st} arrival

T_2 = time from T_1 to 2^{nd} arrival

T_3 = " " T_2 to 3^{rd} arrival

$\vdots$

$\vdots$

T_k = time from T_{k-1} to k^{th} arrival

Then let

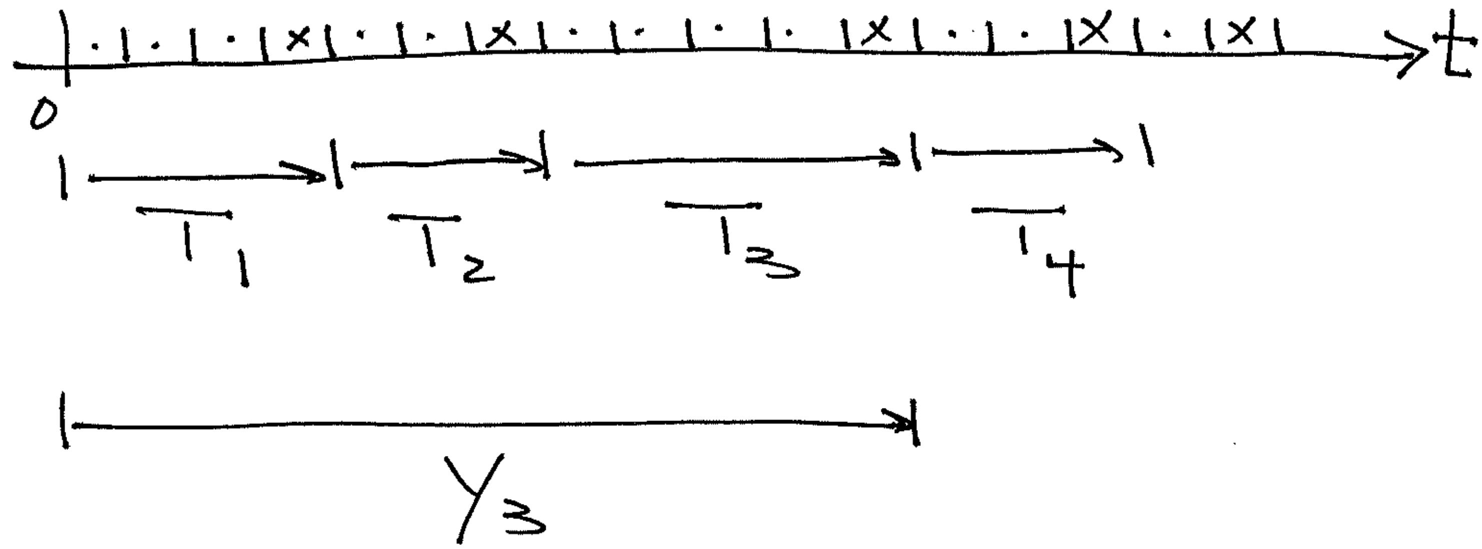
$$Y_k = T_1 + T_2 + \cdots + T_k$$

$$= \text{time to } k^{th} \text{ arrival.}$$

Altenately we could define

$$T_1 = Y_1$$

$$T_k = Y_k - Y_{k-1} \quad (k = 2, 3, \ldots)$$



What are distributions of $Y_2, Y_3, \ldots$

<u>note</u>: $Y_k = Y_{k-1} + T_k \quad (k=2,3,\ldots)$

Thus
$$Y_2 = Y_1 + T_2$$
$$Y_3 = Y_2 + T_3$$
$$\vdots$$

use discrete convolution product to find PMF $P_{Y_k}(t)$.

Exercise

Do this, and prove your general formula by induction on K.

Alternate Solution: use independence

$$P_{Y_K}(t) = P(Y_K = t)$$

$$= P\big((K-1) \text{ arrivals in } [1, t-1],\ 1 \text{ arrival at time } t\big)$$

$$= \underbrace{P\big((K-1) \text{ arrivals in } [1, t-1]\big)}_{\text{Binomial}(p,\, t-1;\, K-1)} \cdot \underbrace{P(\text{arrival at } t)}_{p}$$

Thus

$$P_{Y_K}(t) = \binom{t-1}{K-1} p^{K-1} (1-p)^{(t-1)-(K-1)} \cdot p$$

$$= \begin{cases} \binom{t-1}{K-1} p^K (1-p)^{t-K} & (t = K, K+1, \dots) \\ 0 & \text{otherwise} \end{cases}$$

We call this <u>Pascal PMF of order K.</u>

Also note

$$E[Y_K] = E[T_1 + \dots + T_K]$$

$$= \frac{1}{p} + \dots + \frac{1}{p} = \boxed{\frac{K}{p}}$$

and

$$Var(Y_K) = Var(T_1) + \dots + Var(T_K) \quad \Big\{ \text{by indep.}$$

$$= \frac{1-p}{p^2} + \dots + \frac{1-p}{p^2} = \boxed{\frac{K(1-p)}{p^2}}$$

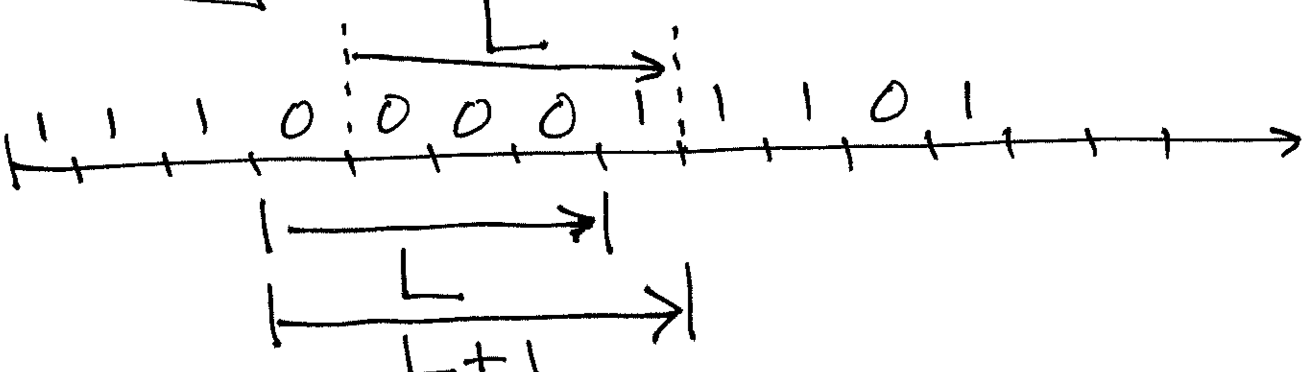
Exercise:

check that

$$\sum_{t=k}^{\infty} P_{Y_k}(t) = 1$$

Ex.

You buy a lottery ticket everyday.
Let L be the length of your 1st losing streak.



What is the dist. of L.

You might think that $L+1$ is geometric, hence L is shifted geometric.

But L cannot be 0, so $L+1$ cannot be 1, so $L+1$ cannot be geometric.

However, L is also time from 2^{nd} loss until next win.

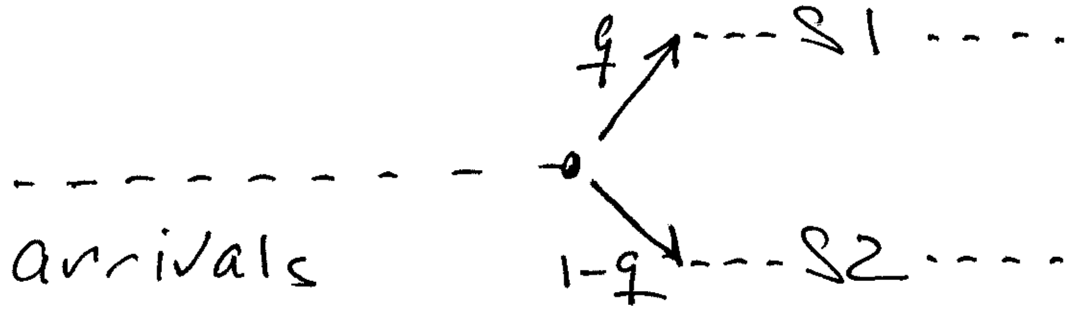
Thus L is Geometric(p)

Splitting a Bernoulli Process

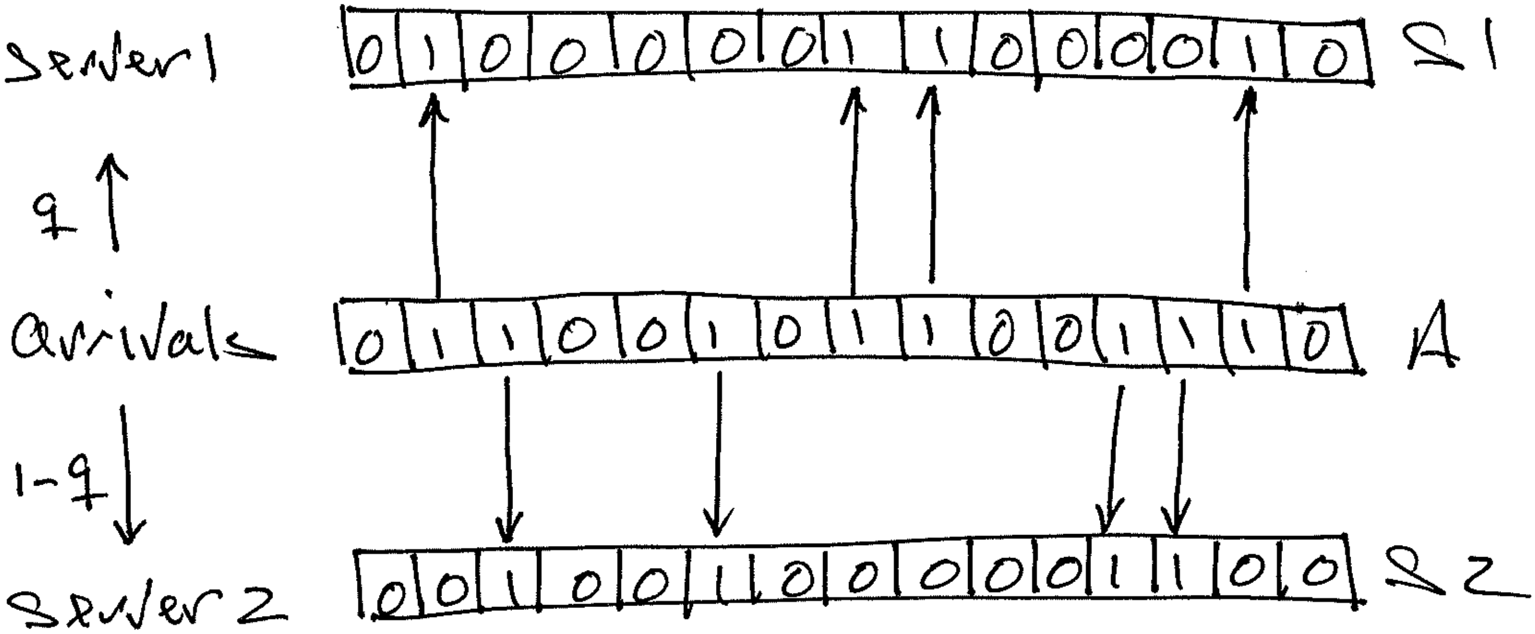
We have a Bernoulli Process (Param. p) producing a stream of arrivals of jobs into a queue. Send each job to one of 2 servers with

$$P(S1) = q$$

$$P(S2) = 1 - q$$



What kind of Processes are seen by S1 and S2 ?



Assume the S1/S2 coin is independent in A. Then

$$P(\text{arrival in } S1) = p \cdot q$$

and

$$P(\text{arrival in } S2) = p(1-q)$$

since all decisions (job/no job, S1/S2) are indep. at different times, both S1 & S2 are Bernoulli.

Thus S1 is Bernoulli with Param. pq, and S2 is Bernoulli with Param $p(1-q)$

Merging two Bernoulli Processes

We're given simultaneous, independent Bernoulli Processes

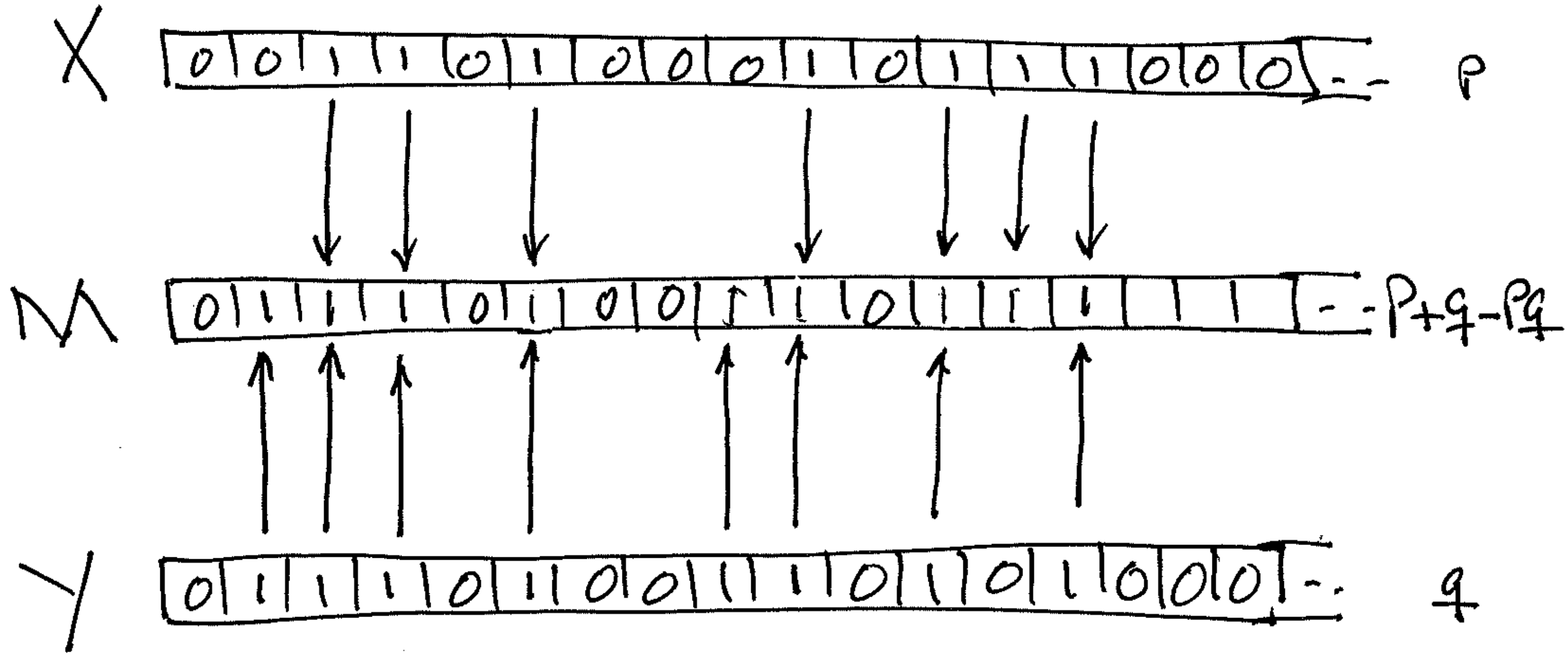
		Param.
stream 1:	$X_1, X_2, \ldots$	p
stream 2:	$Y_1, Y_2, \ldots$	q
Merge Streams:	$M_1, M_2, \ldots$	

according to following rule.

$$M_t = \begin{cases} 1 & \text{if either } X_t = 1 \text{ or } Y_t = 1 \\ 0 & \text{otherwise} \end{cases}$$

What kind of arrival processes is M?

Param



observe

$$P(M_t = 1) = 1 - P(M_t = 0)$$

$$= 1 - P(X_t = 0, Y_t = 0)$$

$$= 1 - P(X_t = 0) \cdot P(Y_t = 0) \quad \text{by indep.}$$

$$= 1 - (1-p)(1-q)$$

$$= 1 - (1 - p - q + pq)$$

$$= p + q - pq$$

To Check M is Bernoulli, we check

$$P(M_{t_1} = 1, M_{t_2} = 1) = P(M_{t_1} = 1) \cdot P(M_{t_2} = 1)$$

True . . .