CSE 107 5-22-25

4.5 Sum of a random # of r.v.s

Let $X_1, X_2, \ldots$ be a sequence of iid random variables, with common PDF $f_X(x)$ (or PMF $p_X(x)$). Let N be a random variable with range$(N) = \{0,1,2\ldots\}$

Assume

$$\{N, X_1, X_2, \ldots\}$$

are independent. Define

$$Y = X_1 + X_2 + \cdots + X_N .$$

Let $E[X], Var(X)$ be the common mean and variance of X_i.

<u>Goal</u>: find $E[Y]$, $Var(Y)$

let $n \geq 0$ be fixed. Then

$$E[Y \mid N=n]$$

$$= E[X_1 + \cdots + X_N \mid N=n]$$

$$= E[X_1 + \cdots + X_n \mid N=n]$$

$$= E[X_1 + \cdots + X_n] \leftarrow \left\{\begin{array}{l}\text{by}\\ \text{indep.}\end{array}\right.$$

$$= E[X_1] + \cdots + E[X_n]$$

$$= n\,E[X]$$

Thus

$$E[Y \mid N] = N \cdot E[X]$$

By law of iterated expectation

$$E[Y] = E[E[Y|N]]$$
$$= E[N \cdot E[X]]$$
$$= E[N] \cdot E[X]$$

Also

$$\mathrm{Var}(Y|N=n)$$
$$= \mathrm{Var}(X_1 + \cdots + X_N \mid N=n)$$
$$= \mathrm{Var}(X_1 + \cdots + X_n \mid N=n)$$
$$= \mathrm{Var}(X_1 + \cdots + X_n) \quad \Big\{ \text{by indep.}$$

$$= Var(X_1) + \cdots + Var(X_n) \quad \text{by indep.}$$

$$= n \cdot Var(X)$$

Thus

$$Var(Y|N) = N \cdot Var(X)$$

By law of total variance

$$Var(Y) = E[Var(Y|N)] + Var(E[Y|N])$$

$$= E[N \cdot Var(X)] + Var(N \cdot E[X])$$

$$= E[N]\cdot Var(X) + E[X]^2 \cdot Var(N)$$

Ex. have 2 coins

coin 1 : $P(\text{head}) = p$

coin 2 : $P(\text{head}) = q$

Flip ① until 1^{st} head. let N be # of flips.

flip ② N times, count # heads.

let Y = # heads. Then

$$Y = X_1 + X_2 + \cdots + X_N$$

where each X_i is Bernoulli(q)

have

$$E[N] = \frac{1}{p}, \quad Var(N) = \frac{1-p}{p^2}$$

and

$$E[X] = q, \quad Var(X) = q(1-q)$$

so

$$E[Y] = E[N] \cdot E[X] = \boxed{\frac{q}{p}}$$

also

$$Var(Y) = E[N] \cdot Var(X) + E[X]^2 \cdot Var(N)$$

$$= \frac{q(1-q)}{p} + \frac{q^2(1-p)}{p^2}$$

$$= \boxed{\frac{q}{p}\left((1-q) + \frac{q}{p}(1-p)\right)}.$$