CSE 107 5-16-25

Supplemental Lecture

Exercise

let X be cont. uniform on $[a,b]$, and Y be cont. unif on $[c,d]$. Find PDF of

$$Z = \max(X, Y) \quad \left\{\begin{array}{l}\text{Assume} \\ X, Y \text{ are} \\ \text{indep.}\end{array}\right.$$

Rmk

when $b<c$, we have $[a,b]$, $[c,d]$ are disjoint, and $X \leq Y$. In this case $Z = \max(X,Y) = Y$, so $f_Z = f_Y$. Similarly if $a>d$, we have $f_Z = f_X$. So Assume both $c \leq b$ and $a \leq d$, whence $[a,b] \cap [c,d] \neq \phi$.

If X, Y are indep. r.v. and $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, our goal is to find PDF (or PMF) of

$$Z = g(X, Y).$$

special case: $g(x,y) = x+y$.

<u>sums and convolutions</u>

Let X, Y be indep. discrete r.v.s and $Z = X + Y$. Then

$$p_Z(z) = P(Z = z)$$
$$= P(X + Y = z)$$

$$= \sum_{\{(x,y) \mid x+y=z\}} P(X=x, Y=y)$$

$$= \sum_{\{(x, z-x) \mid x \in \mathrm{range}(X)\}} P(X=x, Y=z-x)$$

$$= \sum_{x} P(X=x) \cdot P(Y=z-x) \quad \Big\{ \text{by indep.}$$

$$= \sum_{x} p_X(x) \cdot p_Y(z-x)$$

Thus

$$p_Z(z) = \sum_{x} p_X(x)\, p_Y(z-x)$$

$$= \sum_{y} p_Y(y) \cdot p_X(z-y)$$

Defn

The convolution product of p_X, p_Y is

$$\sum_x p_X(x) \cdot p_Y(z-x) = \sum_y p_Y(y) p_X(z-y)$$

notation: $(p_X * p_Y)(z) = (p_Y * p_X)(z)$

we've shown: $p_Z = p_X * p_Y$, i.e.

$$p_{X+Y} = p_X * p_Y = p_Y * p_X.$$

Let X, Y independent (Jointly) continuous r.v.s, and let

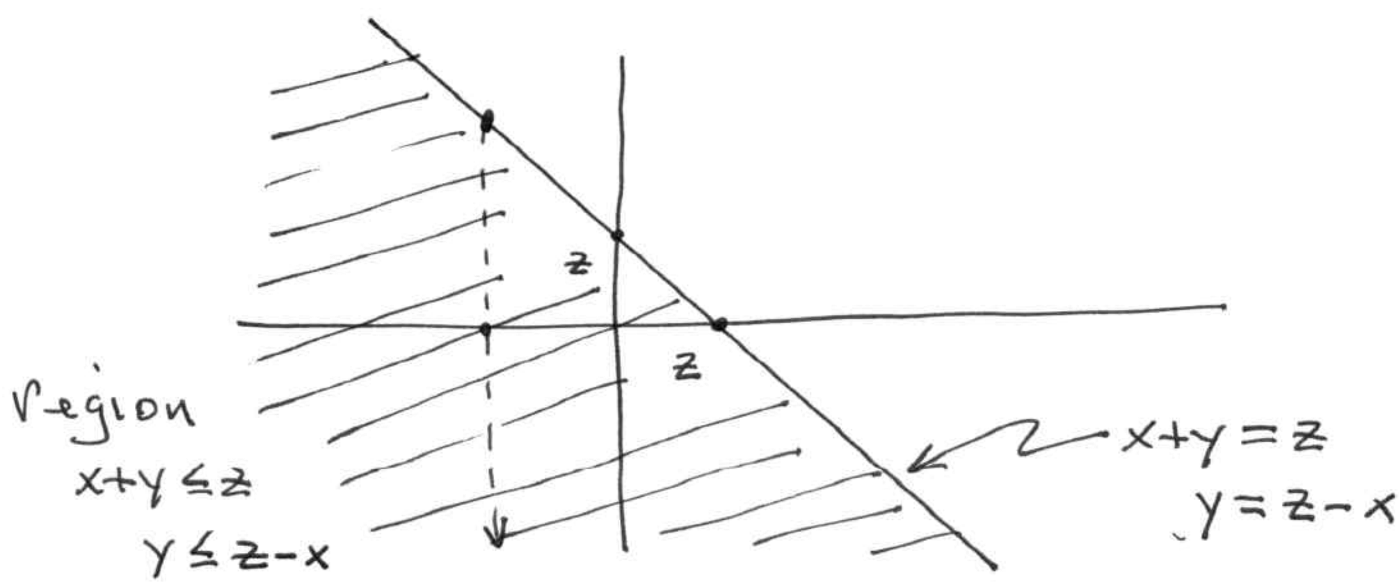
$$Z = X + Y$$

Then

$$F_Z(z) = P(Z \le z)$$

$$= P(X + Y \le z)$$

$$= \iint_{\{(x,y) \mid x+y \le z\}} f_{X,Y}(x,y)\, dy\, dx$$



$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) f_Y(y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \left(\int_{-\infty}^{z-x} f_Y(y)\, dy \right) dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot F_Y(z-x)\ dx$$

Thus

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \frac{d}{dz} \int_{-\infty}^{\infty} f_X(x) \cdot F_Y(z-x)\ dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot \frac{d}{dz} F_Y(z-x)\, dx$$

so

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx$$

also

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) \cdot f_X(z-y)\, dy$$

Defn

The convolution of f_X, f_Y is

$$(f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

Thus

$$f_{X+Y} = f_X * f_Y = f_Y * f_X$$

Rmks

Remember, both discrete and continuous formulas for f_{X+Y} (or P_{X+Y}) require X, Y to be independent.

Ex.

Let X, Y indep. cont. uniform on $[a,b]$. i.e.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

and similarly for $f_Y(y)$. Let

$$Z = X + Y$$

Then

$$f_Z(z) = (f_X * f_Y)(z)$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx$$

The integrand is eq. to $\frac{1}{(b-a)^2}$ iff both

$$a \leq x \leq b \quad \text{and} \quad a \leq z-x \leq b$$

i.e.

$$a \leq x \leq b \quad \text{and} \quad z-b \leq x \leq z-a$$

(and 0 otherwise). This is true iff

$$\max(a, z-b) \leq x \leq \min(b, z-a)$$

Thus

$$f_Z(z) = \int_{\max(a,z-b)}^{\min(b,z-a)} \frac{1}{(b-a)^2}\,dx$$

$$= \frac{\min(b,z-a) - \max(a,z-b)}{(b-a)^2}$$

The interval of integration is empty if either

$$z-a < a \longrightarrow z < 2a$$

or

$$z-b > b \longrightarrow z > 2b$$

so we require $2a \le z \le 2b$

Thus

$$f_Z(z) = \begin{cases} \dfrac{\min(b, z-a) - \max(a, z-b)}{(b-a)^2} & (2a \leq z \leq 2b) \\ 0 & (\text{otherwise}) \end{cases}$$

Ex.

Let X, Y be indep normal r.v.s with means: μ_X, μ_Y and Variances σ_X^2, σ_Y^2, respectively. So

$$f_X(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma_X} \cdot e^{-(x-\mu_X)^2 / 2\sigma_X^2}$$

similarly for Y.

Let $Z = X + Y$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_X} \cdot e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_Y} \cdot e^{\frac{-(z-x-\mu_Y)^2}{2\sigma_Y^2}} \, dx$$

$= \ldots$ (exercise) $\ldots$

$$= \frac{1}{\sqrt{2\pi} \cdot \sqrt{\sigma_X^2 + \sigma_Y^2}} \cdot e^{\frac{-(z-(\mu_X+\mu_Y))^2}{2(\sigma_X^2+\sigma_Y^2)}}$$

Thus Z is normal with

Mean: $\mu_X + \mu_Y$

Variance: $\sigma_X^2 + \sigma_Y^2$

It follows that if X, Y are indep. normal r.v.s then

$$aX + bY$$

is also normal.