ese 107 5-15-25

If X, Y are independent

(1) $E[X \cdot Y] = E[X] \cdot E[Y]$

<u>Exercise</u> Prove this (easy)

(2) $E[g(X) \cdot h(Y)] = E[g(X)] E[h(Y)]$

<u>Exercise</u>
Prove that X, Y indep. implies $g(X), h(Y)$ are indep. (hard)

<u>Hint</u>: adapt proof in discrete case #44, ch. 2, p. 134.

(3) $Var(X+Y) = Var(X) + Var(Y)$

<u>Exercise</u> Prove this (easy)

<u>3.6 Baye's Rule</u>

original version:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

If $A_1, A_2, \ldots, A_n$ form a partition of Ω, by total probability

$$P(B) = \sum_{i=1}^{n} P(A_i \cap B)$$

$$= \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)$$

so for any A_k $(k = 1, \ldots, n)$

$$\boxed{P(A_k|B) = \frac{P(B|A_k) \cdot P(A_k)}{\sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)}}$$

If X, Y are discrete r.v.s

$$\boxed{p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \cdot p_X(x)}{\sum_t p_{Y|X}(y|t) \cdot p_X(t)}}$$

If X, Y are cont. r.v.s

$$\boxed{f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)\, f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|t)\, f_X(t)\, dt}}$$

Ex.

Let Y be normal with $\sigma = 1$,

and $\mu = X$, another r.v..

suppose X is continuous uniform

on $[1, 3]$.

a.) find $f_Y(y)$

we are given

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \cdot e^{-(y-x)^2/2} & 1 \leq x \leq 3 \\ & (y \in \mathbb{R}) \\ 0 & \text{otherwise} \\ & (y \in \mathbb{R}) \end{cases}$$

so

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_{-\infty}^{\infty} f_{Y|X}(y|x) \cdot f_X(x)\, dx$$

$$= \int_1^3 \frac{1}{2} \cdot \frac{e^{-(x-y)^2/2}}{\sqrt{2\pi}}\, dx$$

sub.
$u = x - y$
$du = dx$
$x = 1 \rightarrow u = 1 - y$
$x = 3 \Rightarrow u = 3 - y$

$\square$

$$= \frac{1}{2}\int_{1-y}^{3-y} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du$$

$$= \boxed{\frac{1}{2}\left(\Phi(3-y) - \Phi(1-y)\right)}$$

b.) find $f_{X|Y}(x|y)$.

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)\cdot f_X(x)}{f_Y(y)}$$

$$= \frac{\cancel{\tfrac{1}{2}}\cdot \frac{1}{\sqrt{2\pi}}\cdot e^{-(y-x)^2/2}}{\cancel{\tfrac{1}{2}}\left(\Phi(3-y) - \Phi(1-y)\right)}$$

$$= \begin{cases} \dfrac{\frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2}}{\Phi(3-y) - \Phi(1-y)} & \begin{array}{l} 1 \le x \le 3 \\ y \in \mathbb{R} \end{array} \\ \\ 0 & \begin{array}{l} \text{otherwise} \\ y \in \mathbb{R} \end{array} \end{cases}$$

e.) suppose we sample Y and get $y=3$. what is probability that $X \le 2$, i.e. find

$$P(X \le 2 \mid Y=3)$$

(exercise) $\boxed{\text{ans} = .2848}$

d.) find $E[Y]$

$\boxed{\text{ans.} = 2}$

4.1 Derived Distributions

If $Y = g(X)$, how to compute

$$f_Y(y)$$

from $f_X(x)$? We call

$f_X(x)$ a <u>derived distribution</u>.

2-step Process

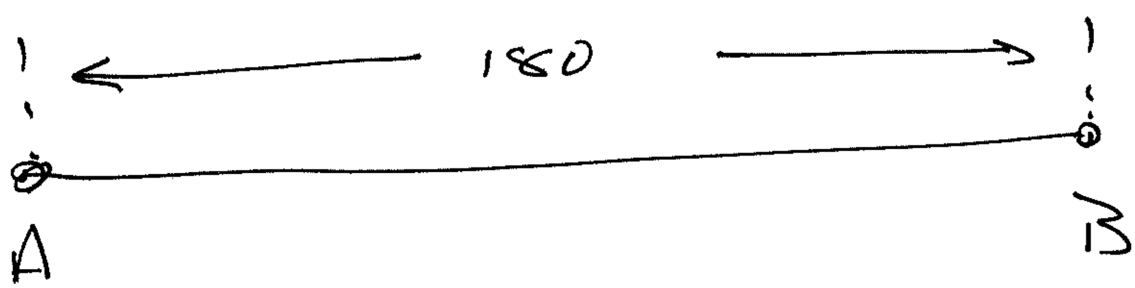
1.) find CDF

$$F_Y(y) = P(g(X) \leq y) = \int_{\{x \mid g(x) \leq y\}} f_X(x)\, dx$$

2.) differentiate

$$f_Y(y) = \frac{d}{dy} F_Y(y) .$$

EX A car travels a dist. of 180 miles at const. rate R, where R is continuous uniform on $30 \le r \le 60$ (mph)



let T be time of travel. find $f_T(t)$.

<u>solution</u>

From $180 = R \cdot T$, we get

$$T = \frac{180}{R}$$

(1) $$F_T(t) = P(T \le t)$$

$$= P\left(\frac{180}{R} \le t\right)$$

$$= P\left(R \ge \frac{180}{t}\right)$$

$$= 1 - P\left(R < \frac{180}{t}\right)$$

$$= 1 - F_R\left(\frac{180}{t}\right)$$

② differentiate

$$f_T(t) = + f_R\left(\frac{180}{t}\right)\left(+\frac{180}{t^2}\right)$$

We have

$$f_R(r) = \begin{cases} \frac{1}{30} & 30 \le r \le 60 \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$f_T(t) = \begin{cases} \frac{1}{30} \cdot \frac{180}{t^2} & 30 \le \frac{180}{t} \le 60 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{6}{t^2} & \text{if } 3 \le t \le 6 \\ 0 & \text{otherwise} \end{cases}$$

$(a \neq 0)$

If $Y = aX + b$, then

$$\boxed{f_Y(y) = \frac{1}{|a|} \cdot f_X\left(\frac{y-b}{a}\right)} \quad a \neq 0$$

Proof

① $F_Y(y) = P(Y \leq y)$

$$= P(aX + b \leq y)$$

$$= \begin{cases} P\left(X \leq \frac{y-b}{a}\right) & a > 0 \\ P\left(X \geq \frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \begin{cases} P\left(X \leq \frac{y-b}{a}\right) & a > 0 \\ 1 - P\left(X < \frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right) & a>0 \\ 1-F_X\left(\frac{y-b}{a}\right) & a<0 \end{cases}$$

(2) $$f_Y(y) = \begin{cases} f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{a} & a>0 \\ -f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{a} & a<0 \end{cases}$$

$$\therefore \quad f_Y(y) = \frac{1}{|a|}\cdot f_X\left(\frac{y-b}{a}\right) \qquad \blacksquare$$

<u>Corollary</u>

if X is normal with mean μ, variance σ^2, then $Y = aX+b$ is also normal w. mean $a\mu+b$ and variance $a^2\sigma^2$, std. dev. $= |a|\cdot\sigma$

<u>Defn</u>

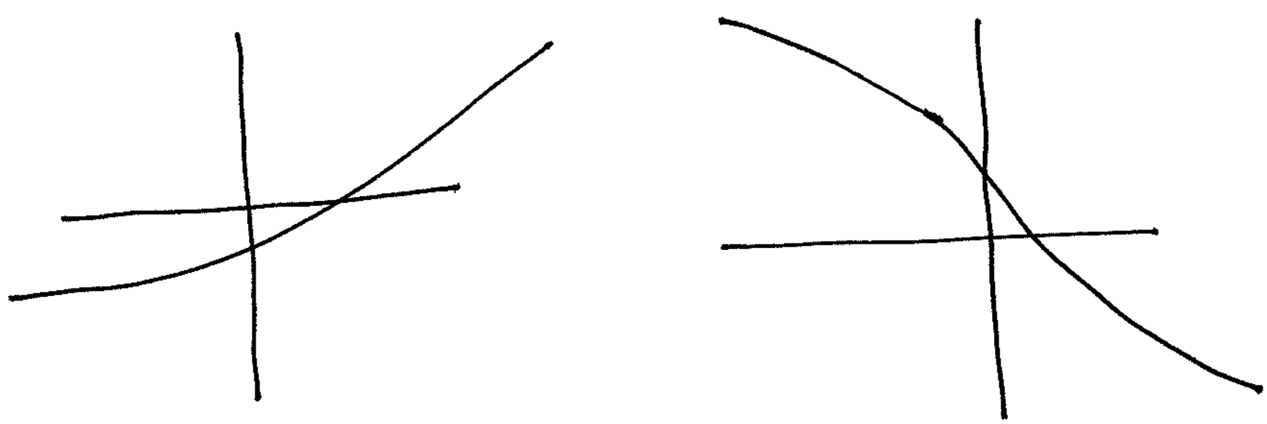
$g : \mathbb{R} \to \mathbb{R}$ is called <u>strictly monotonic</u> iff either

(1) $x_1 < x_2 \longrightarrow g(x_1) < g(x_2)$ $\left\{ \begin{array}{l} \text{strictly} \\ \text{increasing} \end{array} \right.$

or

(2) $x_1 < x_2 \longrightarrow g(x_1) > g(x_2)$ $\left\{ \begin{array}{l} \text{strictly} \\ \text{decreasing} \end{array} \right.$

(1) (2)



<u>note</u>: such functions are invertable

let $h = g^{-1}$

(1) $\iff$ h is strictly increasing

and

(2) $\iff$ h " " decreasing

If g is differentiable then

(1) $\iff g' > 0 \iff h' > 0$

(2) $\iff g' < 0 \iff h' < 0$

Theorem

let $Y = g(X)$, then

$$\boxed{f_Y(y) = f_X(h(y)) \cdot |h'(y)|}$$

Proof

Suppose g is strictly increasing.
(leave decreasing as exercise).

Then

① $F_Y(y) = P(Y \le y)$

$= P(g(X) \le y)$

$= P(X \le h(y))$ { using h is inc.

$= F_X(h(y))$

② $f_Y(y) = f_X(h(y)) \cdot h'(y)$

$= f_X(h(y)) \cdot |h'(y)|$ { since $h' > 0$

■

<u>Ex.</u> Let $Y = X^2$ where X is a cont. r.v. with

$$\text{range}(X) = [0, \infty)$$

Then also

$$\text{range}(Y) = [0, \infty)$$

so $g : [0,\infty) \to [0,\infty)$ is strictly increasing, $g(x) = x^2$ has inverse $h(y) = \sqrt{y}$. Thus

$$f_Y(y) = \begin{cases} f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} & (y \geq 0) \\ 0 & \text{otherwise} \end{cases}$$

let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

let X, Y be <u>indep.</u> r.v.s, both cont uniform on $[0, 1]$

let

$$Z = \overset{g}{\overset{\downarrow}{\max}}(X, Y)$$

We know

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 0 \end{cases}$$

Similarly for Y.

<u>goal</u>: find f_Z

$$F_Z(z) = P(Z \le z)$$
$$= P(\max(X,Y) \le z)$$
$$= P(X \le z, Y \le z)$$
$$= P(X \le z) \cdot P(Y \le z)$$
$$= F_X(z) \cdot F_Y(z)$$
$$= z^2$$

$$\therefore f_Z(z) = \begin{cases} 2z & \text{if } 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$$