CSE 107 5-13-25

Ex show that the geometric r.v. is 'memoryless'.

$$\overbrace{\underbrace{0\ 0\ 0\cdots\cdots 0}_{m}\ \underbrace{0\ 0\cdots\cdots 0\ 0\ 1}_{X}}^{Y}$$

let

Y = # flips to 1st head : Geometric(p)

X = # flips after m^{th} to 1st head.

let $A = \{ Y > m \}$. must show

$$P_Y(k) = P_{X|A}(k) \quad (\text{for } k = 1, 2, 3, \dots)$$

Proof

$$p_{X|A}(k) = P(X=k \mid A)$$

$$= P(Y=m+k \mid Y>m)$$

$$= \frac{P(Y=m+k,\ Y>m)}{P(Y>m)}$$

$$= \frac{P(Y=m+k)}{P(Y>m)}$$

$$= \frac{p(1-p)^{m+k-1}}{(1-p)^m}$$

$$= p\cdot(1-p)^{k-1} = p_Y(k)$$

for $k = 1, 2, 3, \ldots$ ∎

Let $A_1, A_2, \ldots, A_n \subseteq \Omega$ be a Partition of Ω. The total Prob. Thm gives

$$P(X \le x) = \sum_{i=1}^{n} P(A_i) \cdot P(X \le x | A_i)$$

Thus

$$F_X(x) = \sum_{i=1}^{n} P(A_i) \cdot F_{X|A_i}(x)$$

diff. both sides w.r.t x:

$$\boxed{f_X(x) = \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)}$$

<u>Defn</u>

If X, Y are jointly continuous and $f_Y(y) > 0$, the <u>conditional PDF of X given Y</u> is

$$\boxed{f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

observe:

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

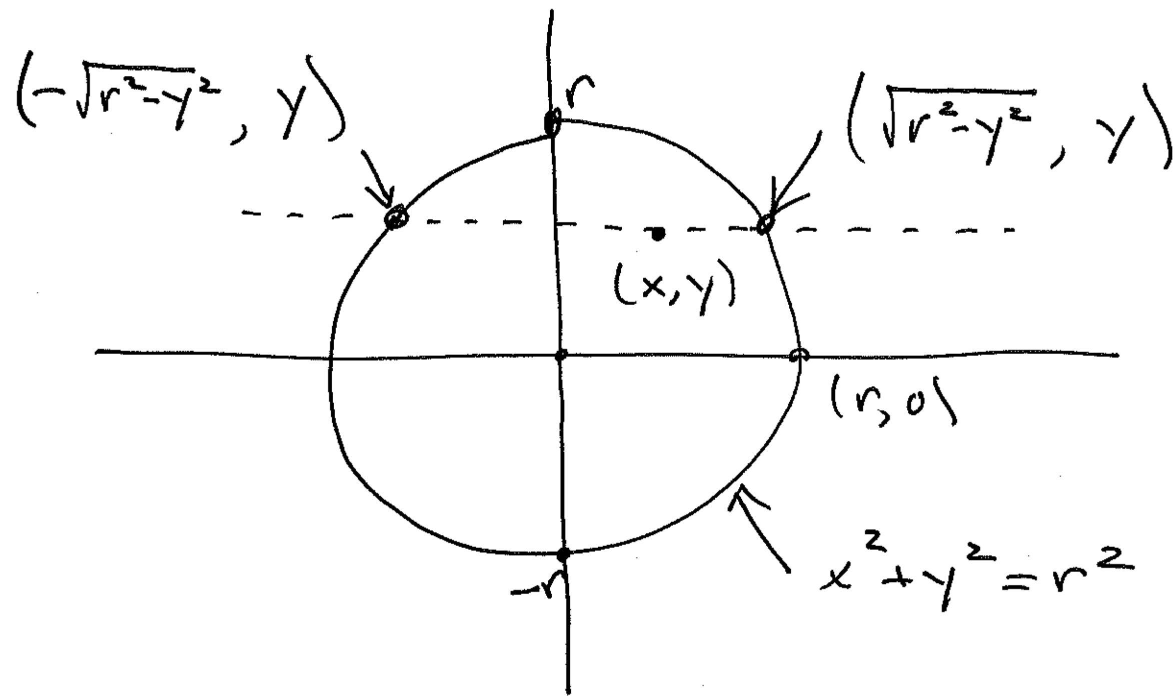
$$= \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}{f_Y(y)}$$

$$= \frac{f_Y(y)}{f_Y(y)} = 1 .$$

<u>EX</u> Throw a dart at circular target of radius r, centered at $(0,0)$. Assume impact point (X,Y) is uniform. Find

$f_{X,Y}(x,y)$, $f_Y(y)$ and $f_{X|Y}(x|y)$.

Solution



We're given

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & \text{if } x^2+y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\, dx$$

$$= \frac{1}{\pi r^2}\left(\sqrt{r^2-y^2} + \sqrt{r^2-y^2}\right)$$

$$= \begin{cases} \dfrac{2\sqrt{r^2-y^2}}{\pi r^2} & \text{if } |y| \le r \\ 0 & \text{otherwise} \end{cases}$$

Finally

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \begin{cases} \dfrac{1}{2\sqrt{r^2-y^2}} & \text{if } x^2+y^2 \le r \\ 0 & \text{otherwise} \end{cases}$$

note: same identities for cond. PDFs as for cond. PMFs.

- $$f_{X|Y}(x|y) \cdot f_Y(y) = f_{X,Y}(x,y)$$
$$= f_{Y|X}(y|x) \cdot f_X(x)$$

Conditional expectation

Let X, Y be jointly cont. r.v.s and $A \subseteq \Omega$ with $P(A) > 0$.

Defns

- $E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)\,dx$
- $E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$

Expected Value rule

- $E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\,dx$
- $E[g(X)\,|\,Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\,dx$

Total Expectation Theorem

- If $A_1, \dots, A_n$ form a partition of Ω, $P(A_i) > 0$ all i, then

$$(1)\quad E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i] \checkmark$$

- similarly

$$(2)\quad E[X] = \int_{-\infty}^{\infty} f_Y(y) \cdot E[X \mid Y=y]\, dy \checkmark$$

<u>note</u>: Partition is $A_y = \{Y=y\}$ for $y \in \text{range}(Y)$.

Proof of (1)

Total Prob. Thm

$$f_X(x) = \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)$$

mult. by x, integrate w.r.t. x:

$$\int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{\infty} x \cdot \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)\, dx$$

$$\therefore \; E[X] = \sum_{i=1}^{n} P(A_i) \cdot \int_{-\infty}^{\infty} x f_{X|A_i}(x)\, dx$$

$$= \sum_{i=1}^{n} P(A_i) \cdot E[X|A_i].$$

Proof of (2)

$$\text{RHS} = \int_{-\infty}^{\infty} E[X|Y=y]\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx \right) \cdot f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y) \cdot f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right) dx$$

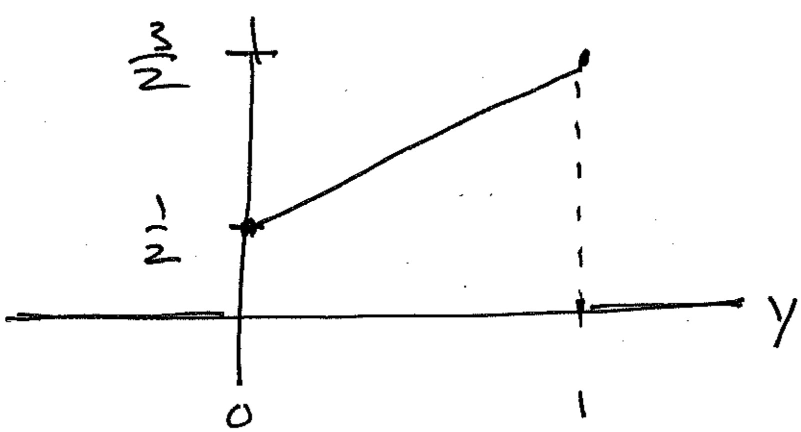
$$= \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= E[X] = LHS. \quad \blacksquare$$

<u>Ex.</u>

Let X, Y be jointly cont. with

$$f_Y(y) = \begin{cases} y + \frac{1}{2} & 0 \le y \le 1 \\ 0 & \text{othwise} \end{cases}$$



and

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{x+y}{y+\frac{1}{2}} & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[X]$.

<u>solution</u>

$$E[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

$$= \int_0^1 x \cdot \left(\frac{x+y}{y+\frac{1}{2}}\right) dx = \frac{1}{y+\frac{1}{2}} \int_0^1 (x^2 + xy)\, dx$$

$$= \frac{1}{y+\frac{1}{2}} \cdot \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 y\right)\Bigg|_0^1 = \frac{\frac{1}{3}+\frac{1}{2}y}{y+\frac{1}{2}}$$

$(0 \le y \le 1)$

so

$$E[X] = \int_{-\infty}^{\infty} E[X \mid Y=y] \cdot f_Y(y)\, dy$$

$$= \int_0^1 \frac{\frac{1}{2}y + \frac{1}{3}}{\cancel{(y+\frac{1}{2})}} \cdot \cancel{(y+\tfrac{1}{2})}\, dy$$

$$= \left(\frac{1}{2} \cdot \frac{1}{2} y^2 + \frac{1}{3} y\right)\Big|_0^1 = \frac{1}{4} + \frac{1}{3} = \boxed{\frac{7}{12}}$$

<u>Exercise</u> Do it the other way

$$E[X] = \int_0^1 \int_0^1 x \underbrace{f_{X,Y}(x,y)}_{f_{X|Y}(x|y) \cdot f_Y(y)}\, dx\, dy$$

Back up:

$$E[X|Y=y] = \begin{cases} \dfrac{\frac{1}{2}y + \frac{1}{3}}{y+\frac{1}{2}} & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

lets compose this fcn. of y with r.v. Y.

$$E[X|Y] = \frac{\frac{1}{2}Y + \frac{1}{3}}{Y + \frac{1}{2}}$$

other notation for (2)

$$\boxed{E[X] = E\left[E[X|Y]\right]}$$

Independence

Let X, Y be jointly continuous.

Defn

We say X, Y are independent

$$\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$$

for all $x \in \text{range}(X)$, $y \in \text{range}(Y)$.

Equivalently:

- $f_{X|Y}(x|y) = f_X(x) \quad (\text{if } f_Y(y) > 0)$

or

- $f_{Y|X}(y|x) = f_Y(y) \quad (\text{if } f_X(x) > 0)$

Can generalize to 3 or more r.v.s

$$f_{X,Y,Z}(x,y,z) = f_X(x) \cdot f_Y(y) \cdot f_Z(z)$$

<u>note</u>: if X, Y are indep. that any $\{X \in I\}$ and $\{Y \in J\}$ are independent. In Particular for $I = (-\infty, x]$, $J = (-\infty, y]$.

Thus

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(s,t)\, ds\, dt$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_X(s) f_Y(t)\, ds\, dt$$

$$= \int_{-\infty}^{x} f_X(s) ds \cdot \int_{-\infty}^{y} f_Y(t) dt$$

$$= P(X \le x) \cdot P(Y \le y)$$

$$= F_X(x) \cdot F_Y(y).$$

<u>note</u> converse is also true.