Exercise: Prove

- $E[aX+b] = aE[X]+b$
- $Var(aX+b) = a^2 Var(X)$

Ex cont. unif. on $[a, b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_a^b x \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{b-a} \cdot \frac{x^2}{2} \Big|_a^b = \frac{1}{b-a} \cdot \frac{1}{2} (b^2 - a^2)$$

$$= \frac{1}{2} \cdot \frac{\cancel{(b-a)}(b+a)}{\cancel{b-a}} = \boxed{\frac{a+b}{2}}$$

$$E[X^2] = \int_a^b x^2 \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{b-a} \cdot \frac{x^3}{3} \Big|_a^b = \frac{1}{3} \cdot \frac{b^3 - a^3}{b-a}$$

$$= \frac{1}{3} \cdot \frac{\cancel{(b-a)} (b^2 + ab + a^2)}{\cancel{(b-a)}}$$

$$= \frac{b^2 + ab + a^2}{3}$$

Hence

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \text{.......} \quad \text{exercise}$$

$$= \boxed{\frac{(b-a)^2}{12}}$$

Exercise

Alice driving time X again

$$f_X(x) = \begin{cases} 1/6 & \text{if } 40 \le x \le 44 \\ 1/18 & \text{if } 44 < x \le 50 \\ 0 & \text{otherwise} \end{cases}$$

compute

- $E[X] = \frac{131}{3}$
- $E[X^2] = \frac{17,228}{9}$
- $Var(X) = \frac{67}{9}$

Verify these.

Exponential Random Variable

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$ is a parameter.

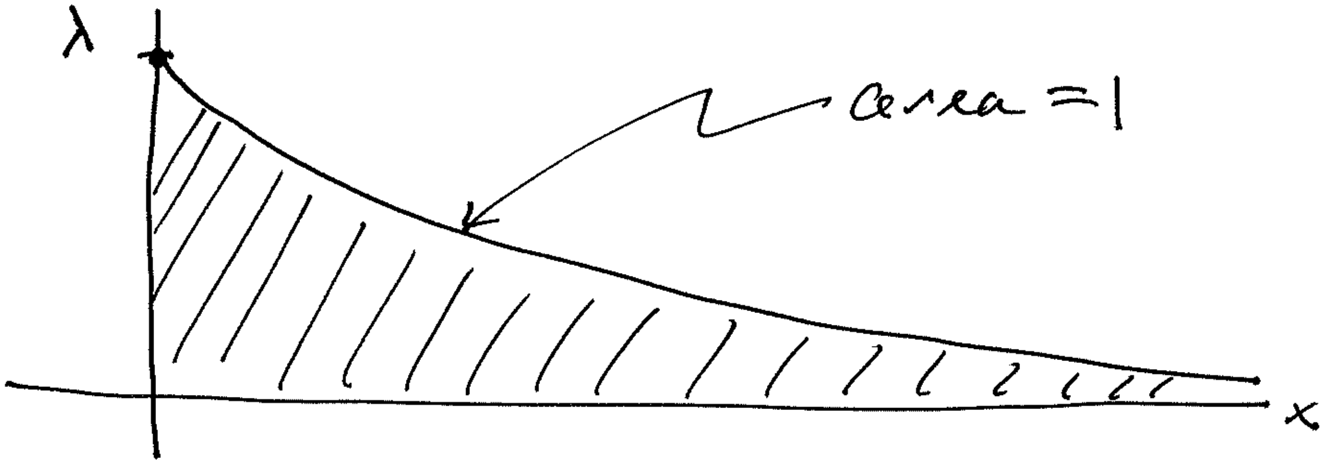
<u>Check!</u>

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \cdot \left(\frac{1}{-\lambda}\right) e^{-\lambda x} \Big|_0^{\infty}$$

$$= -\lim_{a \to \infty} \left(e^{-\lambda a} - 1\right)$$

$$= -(0 - 1) = 1 \quad \checkmark$$



Compute:

$$E[X] = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}$$

- $E[X] = -\int_0^\infty x \cdot (-\lambda e^{-\lambda x}) dx$

int. by Parts: $\begin{cases} u = x & dv = -\lambda e^{-\lambda x} dx \\ du = dx & v = e^{-\lambda x} \end{cases}$

$$= -\left(x e^{-\lambda x} \Big|_0^\infty - \int_0^\infty e^{-\lambda x} dx \right)$$

$$= -\left(0 - \left(\frac{-1}{\lambda}\right) e^{-\lambda x} \Big|_0^\infty \right)$$

$$= -\frac{1}{\lambda} \cdot (0 - 1) = \boxed{\frac{1}{\lambda}}$$

- $E[X^2] = -\int_0^\infty x^2 \left(-\lambda e^{-\lambda x}\right) dx$

int by Parts: $\begin{cases} u = x^2 & dv = -\lambda e^{-\lambda x} dx \\ du = 2x\,dx & v = e^{-\lambda x} \end{cases}$

$= \text{.....} \quad \text{exercise} \quad \text{....}$

$= \frac{2}{\lambda^2}$

- $Var(X) = E[X^2] - E[X]^2$

$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$

$= \boxed{\frac{1}{\lambda^2}}$

Note:

$E[X]$ has units $\frac{\text{time}}{\text{event}}$

$\frac{1}{\lambda}$ " " $\frac{\text{time}}{\text{event}}$

λ " " $\frac{\text{events}}{\text{time}}$

so λ is called the arrival rate.

Lemma

If X is an exponential r.v. then

$$P(X \geq a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \checkmark \\ 1 & \text{if } a \leq 0 \checkmark \end{cases}$$

<u>Proof</u>

let $a > 0$. then

$$P(X \geq a) = \int_a^\infty \lambda e^{-\lambda x}\,dx$$

$$= \lambda \left(-\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_a^\infty$$

$$= -\left(0 - e^{-\lambda a}\right) = e^{-\lambda a}$$

$\blacksquare$

Ex.

The time X until the next earthquake (4.0 or above) is modeled as an exponential r.v. with a mean of 53 minutes (world-wide).

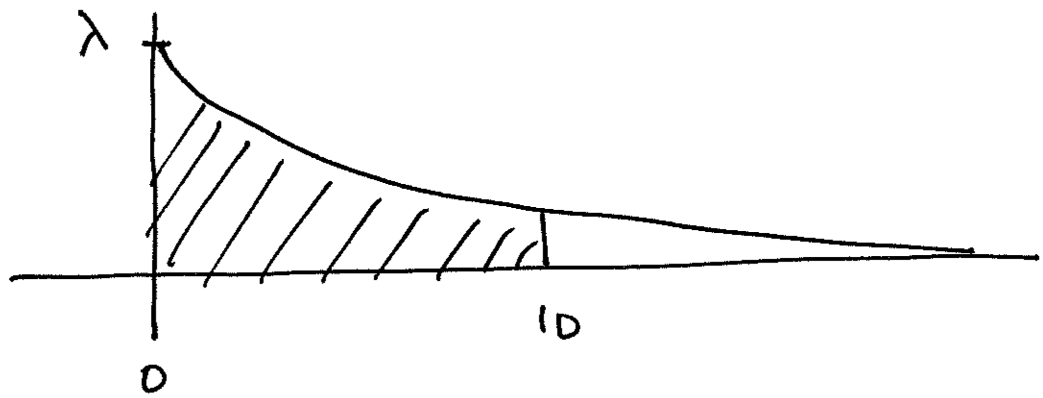
What is the probability that such an earthquake occurs within the next 10 minutes.

Solution

We have $E[X] = \frac{1}{\lambda} = 53$

$\therefore \lambda = \frac{1}{53}$. Thus

$$P(0 \le X \le 10) = P(X \ge 0) - P(X > 10)$$



$$= 1 - e^{-\lambda \cdot 10}$$

$$= 1 - e^{-\frac{10}{53}} = \boxed{0.1719}$$

Exercise

determine $a > 0$ s.t. the prob. of an earthquake in next a minutes is 0.99, i.e. $P(0 \le X \le a) = 0.99$

Solu.

Want $P(0 \leq X \leq a) = 0.99$

$\therefore P(X \geq 0) - P(X > a) = 0.99$

$\therefore \quad 1 - e^{-\lambda a} = .99$

$.01 = e^{-\lambda a}$

$\ln(.01) = -\frac{a}{53}$

$\therefore \quad a = -53 \ln\left(\frac{1}{100}\right) = 53 \ln(100)$

$= 244.07$ min

$= \boxed{4.068 \text{ hrs}}$

3.2 Cumulative Distribution Functions (CDF)

Defn

Let X be a r.v. The cumulative distribution function (CDF) of X is

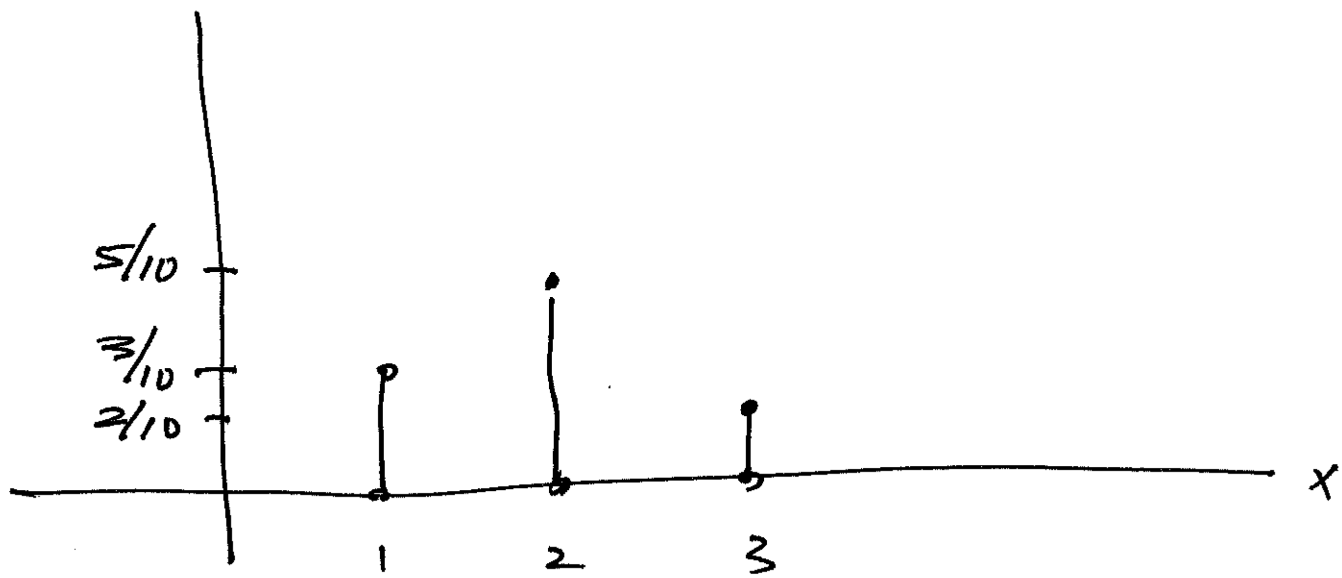
$$\boxed{F_X(x) = P(X \le x)}$$

note

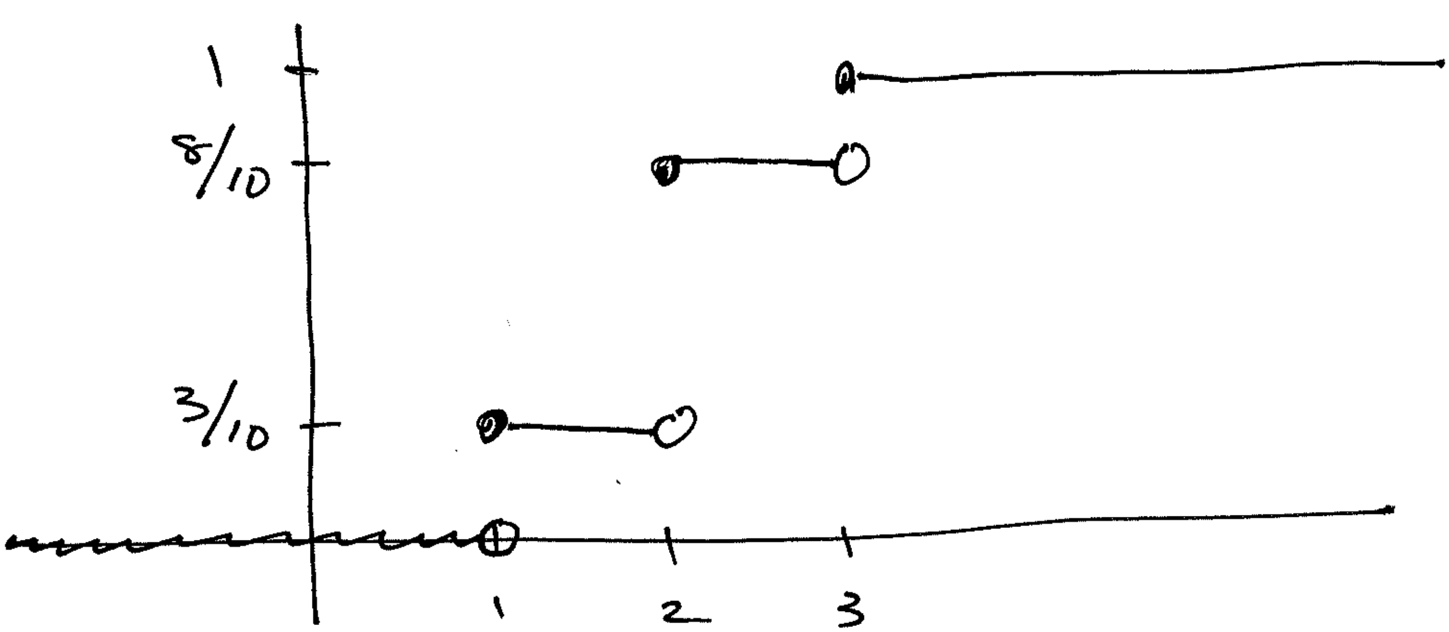
$$F_X(x) = \begin{cases} \sum_{k \le x} p_X(k) & (X \text{ discrete}) \\ \int_{-\infty}^{x} f_X(t)\,dt & (X \text{ continuous}) \end{cases}$$

Ex. (discrete)

$$\text{PMF}: \quad p_X(k) = \begin{cases} 3/10 & \text{if } k=1 \\ 5/10 & \text{if } k=2 \\ 2/10 & \text{if } k=3 \\ 0 & \text{otherwise} \end{cases}$$



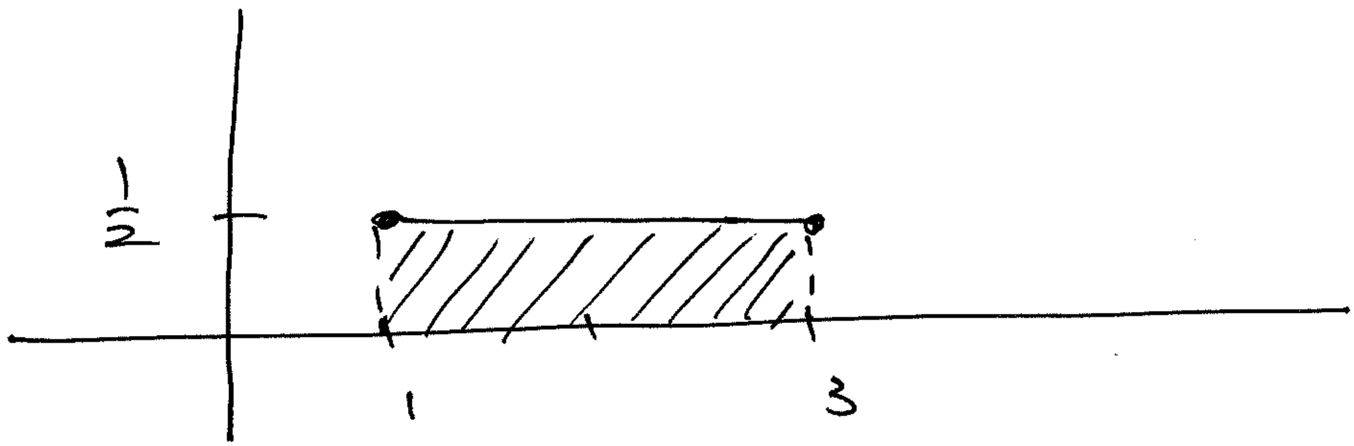
CDF $F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ 3/10 & \text{if } 1 \leq x < 2 \\ 8/10 & \text{if } 2 \leq x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$



Ex. (continuous)

uniform on $[1, 3]$

PDF: $f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$

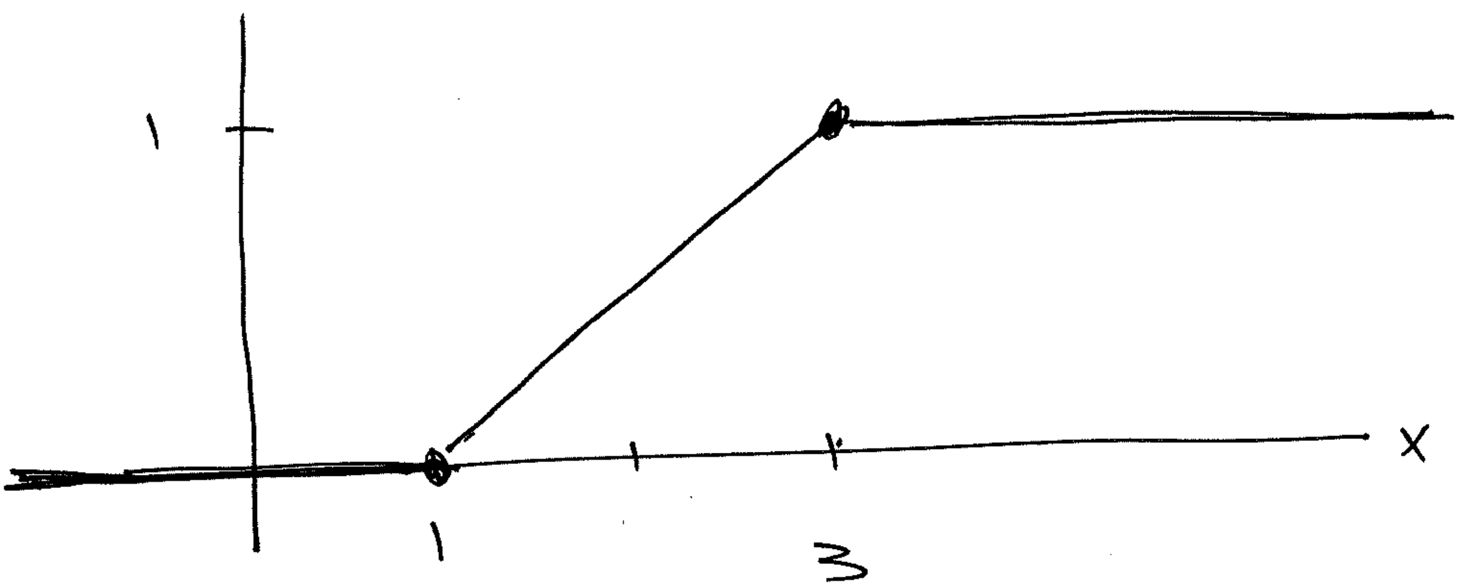


CDF: $F_X(x) = \int_1^x \frac{1}{2}\,dt = \frac{1}{2}t\Big|_1^x = \frac{x-1}{2}$

(if $1 \le x \le 3$)

so for all $x \in \mathbb{R}$

$$F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{x-1}{2} & \text{if } 1 \leq x \leq 3 \\ 1 & \text{if } x > 3 \end{cases}$$



- note the CDF

$$F_X(x) = P(X \le x)$$

is monotonically non-decreasing.

$$x_1 \le x_2 \implies F_X(x_1) \le F_X(x_2)$$

- if X is discrete, then F_X is piecewise constant

$$F_X(k) = \sum_{i \le k} p_X(i)$$

and

$$p_X(k) = P(X=k)$$
$$= P(X \le k) - P(X \le k-1)$$
$$= F_X(k) - F_X(k-1)$$

(this assumes range$(X) \subseteq \mathbb{Z}$.)

- If X is continuous, then $F_X(x)$ is a continuous function

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

and

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Ex.

You can take a test 3 times. Your final score X is the maximum of the 3 scores

$$X = \max(X_1, X_2, X_3)$$

where X_i is score on i^{th} test.

Assume scores on different tests are independent, uniformly distributed over $\{1, 2, \dots, 10\}$

$$P_{X_i}(k) = \begin{cases} 1/10 & \text{if } k = 1, 2, \dots, 10 \\ 0 & \text{otherwise} \end{cases}$$

Find the PMF of X.

<u>solution</u>

first find CDF $F_X(k)$, then difference to find

$$P_X(k) = F_X(k) - F_X(k-1)$$

so

$$F_X(k) = P(X \le k)$$

$$= P(X_1 \le k, X_2 \le k, X_3 \le k)$$

$$= P(X_1 \le k) \cdot P(X_2 \le k) \cdot P(X_3 \le k)$$

$$= F_{X_1}(k) \cdot F_{X_2}(k) \cdot F_{X_3}(k)$$

$$= \left(\frac{K}{10}\right) \cdot \left(\frac{K}{10}\right) \cdot \left(\frac{K}{10}\right)$$

$$= \left(\frac{K}{10}\right)^3$$

$$\therefore P_X(K) = \begin{cases} \left(\frac{K}{10}\right)^3 - \left(\frac{K-1}{10}\right)^3 & \text{if } K = 1, \dots, 10 \\ 0 & \text{otherwise} \end{cases}$$