hw1 #2

let $p_i = P(\text{winning against Player } i)$ $(i=1,2,3)$. Assume $p_1 > p_2 > p_3$.

let

$A = \{\text{win against } 1\}$, $P(A) = p_1$

$B = \ldots\ 2 \ldots\ p_2$

$C = \ldots\ 3 \ldots\ p_3$

$E = \{\text{win tournament}\}$.

say play in order: $\boxed{123}$

Then $E = (A \cap B) \cup (B \cap C)$

$$P(E) = P\big((A\cap B) \cup (B \cap C)\big) \quad \Big\{ \text{use inclusion-exclusion}$$

$\vdots$

$$P(A\cap B) = P(A) \cdot \underbrace{P(B|A)}_{\substack{\text{assume} \\ = P(B)}} = P(A)\cdot P(B) = P_1 \cdot P_2$$

$\Big\{$ called independence.

$$P(B\cap C) = P(B) \cdot \underbrace{P(C|B)}_{P(C)} = P_2 \cdot P_3$$

$$P(A\cap B\cap C) = P(A) \cdot \underbrace{P(B|A)}_{P(B)} \cdot \underbrace{P(C|B\cap A)}_{P(C)} = P_1 P_2 P_3$$

1.5 Independence

Recall: cond. Probability

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

Its possible that

(*) $P(A|B) = P(A)$

We say in this case! A, B are independent.

(**) $\boxed{P(A\cap B) = P(A)\cdot P(B)}$

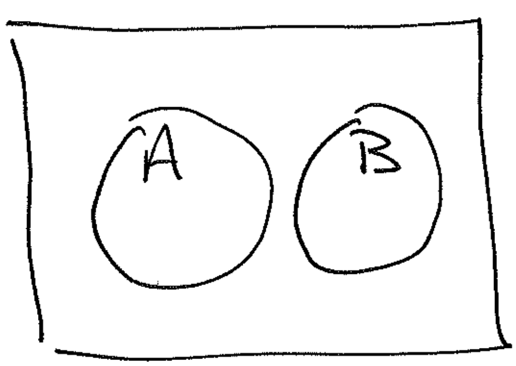
Defn

$A, B \subseteq \Omega$ are <u>independent</u> iff $**$ holds.

<u>note</u>: independence is a symmetric relation on pairs of events.

<u>note</u>

$$A \cap B = \emptyset \;\not\Longleftrightarrow\; A, B \text{ independent}$$



Ex. Throw two fair dice.

$$A_i = \{1^{st} \text{ die is } i\} \quad 1 \le i \le 6$$

$$B_j = \{2^{nd} \text{ die is } j\} \quad 1 \le j \le 6$$

$$P(A_i) = \frac{6}{36} = \frac{1}{6}$$

$$P(B_j) = \frac{6}{36} = \frac{1}{6}$$

If A_i, B_j are independent, then

$$P((i,j)) = P(A_i \cap B_j) = P(A_i) \cdot P(B_j)$$

$$= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

So Independence $\Rightarrow$ Discrete Uniform Prob. law.

The converse is also true. ($\Leftarrow$)

let $C = \{\text{sum} = 5\}$

$= \{14, \underline{23}, 32, 41\}$

$\therefore P(C) = \frac{4}{36} = \frac{1}{9}$

- Is C independent of A_2?

$A_2 = \{21, \underline{\underline{22}}, \underline{23}, 24, 25, 26\}$

$A_2 \cap C = \{23\}$

$\therefore P(A_2 \cap C) = \frac{1}{36}$

$P(A_2) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{9} = \frac{1}{54} \neq \frac{1}{36}$

$\therefore$ A_2, C are __not__ independent.

- let $D = \{\text{doubles}\}$

$$= \{11, 22, 33, 44, 55, 66\}$$

$$\therefore P(D) = \frac{6}{36} = \frac{1}{6}$$

$$D \cap A_2 = \{22\}$$

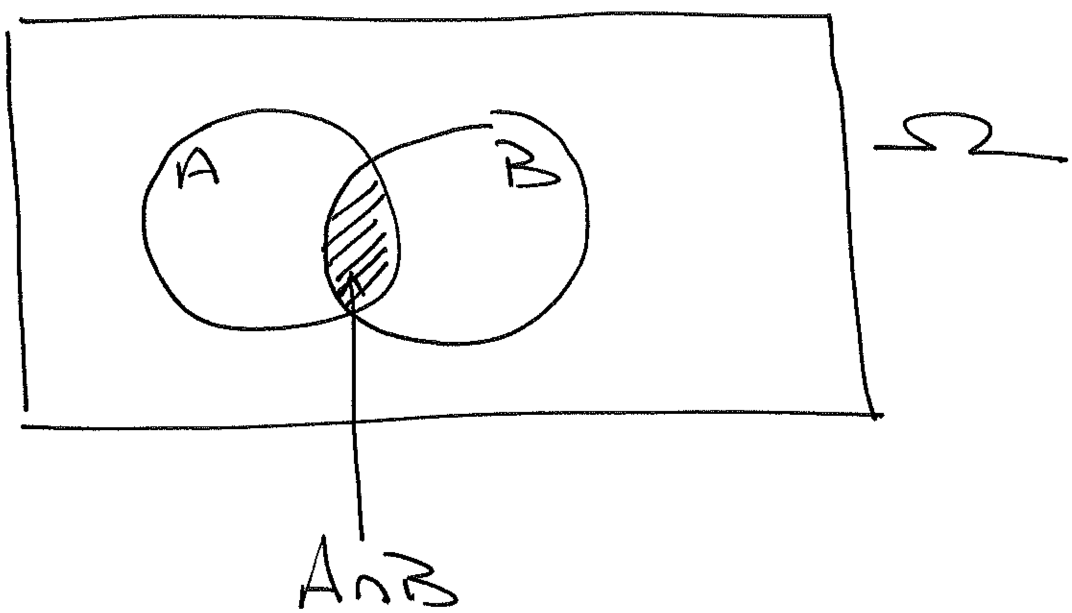
$$\therefore P(D \cap A_2) = \frac{1}{36}$$

$$P(D) \cdot P(A_2) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

Thus A_2, D <u>are</u> independent.

Geometric Picture

Consider $P(A) \sim \text{area}(A)$



$$\frac{\text{area}(A \cap B)}{\text{area}(B)} = \frac{\text{area}(A)}{\text{area}(\Omega)}$$

$$P(A|B) = P(A)$$

Exercise

Let $A, B \subseteq \Omega$. Prove that the following statements are equivalent.

(1) A, B are independent

(2) A, B^c " "

(3) A^c, B " "

(4) A^c, B^c " "

Conditional Independence

Recall, if $P(C) > 0$, then

$$P(\cdot \mid C)$$

is a valid Prob. law. Thus we can define indep. w.r.t. $P(\cdot \mid C)$.

Defn

We say A, B are <u>conditionally independent given C</u> iff

$$\boxed{P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)}.$$

Note:

$$P(A\cap B|C) = \frac{P(A\cap B\cap C)}{P(C)}$$

$$= \frac{\cancel{P(C)}\cdot P(B|C)\cdot P(A|B\cap C)}{\cancel{P(C)}}$$

$\uparrow$ by mult. theorem

$$= P(B|C)\cdot P(A|B\cap C)$$

Alternate definition of cond. indep. of A, B given C.

$$\cancel{P(B|C)}\cdot P(A|B\cap C) = P(A|C)\cdot \cancel{P(B|C)}$$

$$\boxed{P(A|B\cap C) = P(A|C)}$$

note:

$$P(A|B|C) = \frac{P(A\cap B|C)}{P(B|C)}$$

$$= \frac{P(A\cap B\cap C)/\cancel{P(C)}}{P(B\cap C)/\cancel{P(C)}}$$

$$= \frac{P(A\cap B\cap C)}{P(B\cap C)}$$

$$= P(A|B\cap C)$$

Warning:

$$\boxed{\text{cond. ind. of } A,B \text{ given } C} \;\begin{array}{c}\not\Longrightarrow \\ \not\Longleftarrow\end{array}\; \boxed{\text{unconditional indep. of } A,B}$$

See examples 1.20, 1.21 on p. 37

Independence of Multiple events

Let $A, B, C \subseteq \Omega$.

<u>Defn</u>

We say $\{A, B, C\}$ is independent iff the following hold

(1) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

$$\begin{cases} (2)\ P(A \cap B) = P(A) \cdot P(B) \\ (3)\ P(A \cap C) = P(A) \cdot P(C) \\ (4)\ P(B \cap C) = P(B) \cdot P(C) \end{cases}$$

(2) & (3) & (4) is called <u>Pairwise Independence</u>

Warning

$$\boxed{\begin{array}{l}\text{Pairwise}\\ \text{indep. of}\\ A, B, C\\ \text{i.e. (2), (3), (4)}\end{array}} \quad \begin{array}{c}\not\Rightarrow\\ \not\Leftarrow\end{array} \quad \boxed{\begin{array}{l}\text{Property (1)}\\ P(A\cap B\cap C) =\\ P(A)\cdot P(B)\cdot P(C)\end{array}}$$

see examples 1.22, 1.23 on p. 39

Binomial Probabilities

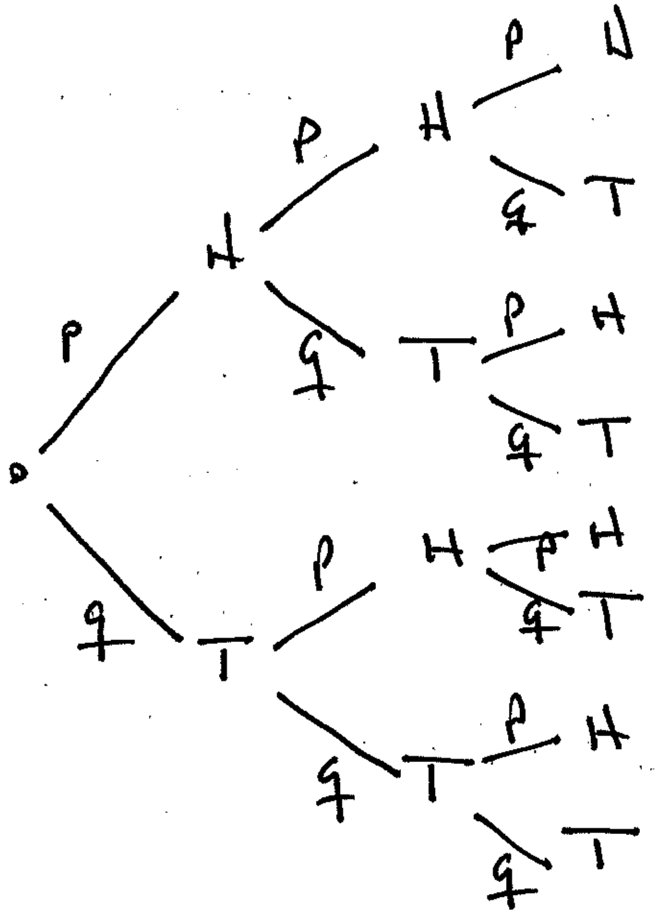
Defn

A simple experiment with exactly 2 outcomes is called a Bernoulli trial.

$$\text{outcomes} = \begin{cases} \text{success} = 1 = \text{heads} \\ \text{failure} = 0 = \text{tails} \end{cases}$$

Ex. Perform a sequence of 3 independent tosses of a coin with

$$P(H) = p \,, \quad P(T) = 1-p = q$$



outcomes	Prob.
HHH	p^3
HHT	p^2q
HTH	p^2q
HTT	pq^2
THH	p^2q
THT	pq^2
TTH	pq^2
TTT	q^3

note:

$$\underbrace{p^3}_{\substack{\uparrow\\ \text{3 heads}}} + \underbrace{3p^2q}_{\substack{\uparrow\\ \text{2 heads}\\ \text{1 tail}}} + \underbrace{3pq^2}_{\substack{\uparrow\\ \text{1 head}\\ \text{2 tails}}} + \underbrace{q^3}_{\substack{\uparrow\\ \text{3 tails}}} = (p+q)^3$$

by Binomial theorem

so

$$P(\#\text{heads} = 2) = 3p^2q$$

In general, if we do n indep. flips of weighted coin

$$\boxed{P(\#\text{heads} = k) = \binom{n}{k} p^k (1-p)^{n-k}}$$

Binomial Probabilities