CSE 107 4-4-25

Supplemental Lecture

We've shown that

$$P(\cdot \mid B)$$

is a valid probability law. So, for instance

$$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 \cap A_2 \mid B)$$

The definition

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

gives $\boxed{P(A \cap B) = P(A \mid B) \cdot P(B)}$

also $$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

given $$\boxed{P(A\cap B) = P(B|A)\cdot P(A)}$$

Thus $$\boxed{P(A|B)\cdot P(B) = P(B|A)\cdot P(A)}$$

Ex.

A test for a certain disease is given to a subject. If subject has the disease, the test returns a positive result with probability .99. If subject does not have disease, then test is positive with prob. .10. The probability that a random individual has disease is .05.

- what is probability of a false positive? (pos. test, no disease)
- What is probability of a false negative? (neg. test, has disease)

Let

$T = \{\text{test is positive}\}$

$D = \{\text{subject has disease}\}$

We are given

$P(T|D) = .99 \quad \therefore P(T^c|D) = .01$

$P(T|D^c) = .10 \quad \therefore P(T^c|D^c) = .90$

$P(D) = .05 \quad \therefore P(D^c) = .95$

we must find

$P(T \cap D^c)$ (false positive)

$P(T^c \cap D)$ (false negative)

we have

$$P(T \cap D^c) = P(T \mid D^c) \cdot P(D^c)$$

$$= (.10) \cdot (.95) = \boxed{.095}$$

and

$$P(T^c \cap D) = P(T^c \mid D) \cdot P(D)$$

$$= (.01)(.05) = \boxed{.0005}$$

Recall: $P(A\cap B) = P(A)\cdot P(B|A)$

<u>Generalize</u>:

$$P(A\cap B\cap C) = P(A)\cdot P(B|A)\cdot P(C|A\cap B)$$

<u>Proof</u>

$$RHS = \cancel{P(A)}\cdot \frac{\cancel{P(A\cap B)}}{\cancel{P(A)}}\cdot \frac{P(A\cap B\cap C)}{\cancel{P(A\cap B)}}$$

$$= P(A\cap B\cap C) = LHS. \quad \blacksquare$$

<u>Further</u>

$$P(A\cap B\cap C\cap D) = P(A)\cdot P(B|A)\cdot P(C|A\cap B) \cdot P(D|A\cap B\cap C)$$

<u>Exercise</u> Prove this

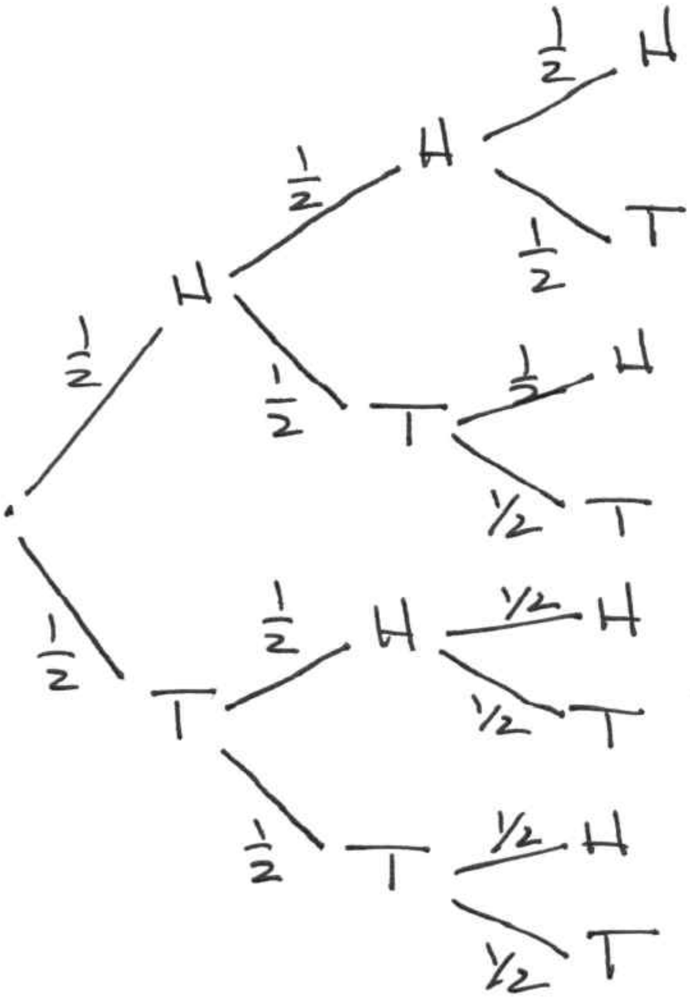
<u>Theorem</u> (multiplication rule)

if we have events $A_1, A_2, \ldots, A_n \subseteq \Omega$

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2) \cdot \ldots \cdot P\left(A_n \middle| \bigcap_{i=1}^{n-1} A_i\right)$$

$$= \prod_{i=1}^{n} P\left(A_i \middle| \bigcap_{j=1}^{i-1} A_j\right)$$

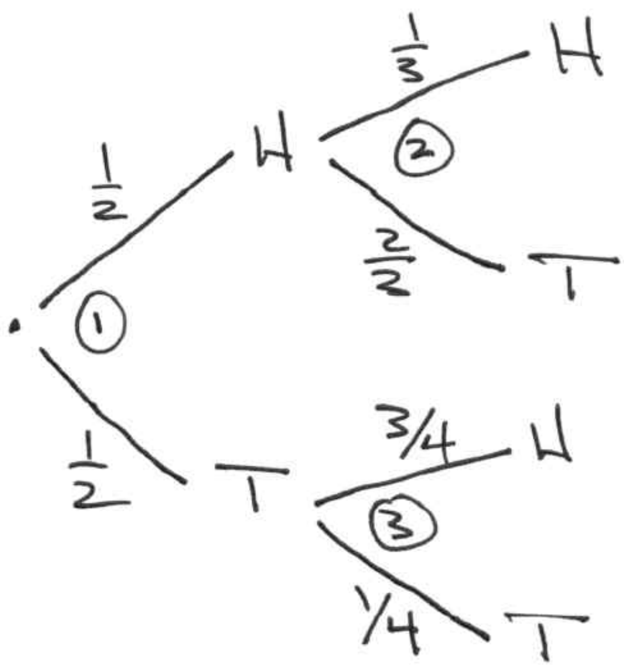
Ex toss 3 fair coins



Outcome	Prob
HHH	$\frac{1}{8}$
HHT	1/8
HTH	1/8
HTT	1/8
THH	1/8
THT	1/8
TTH	1/8
TTT	1/8

We recover the discrete uniform probability law.

Ex. have 3 coins, 2 tosses



HH $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

HT $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$

TH $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$

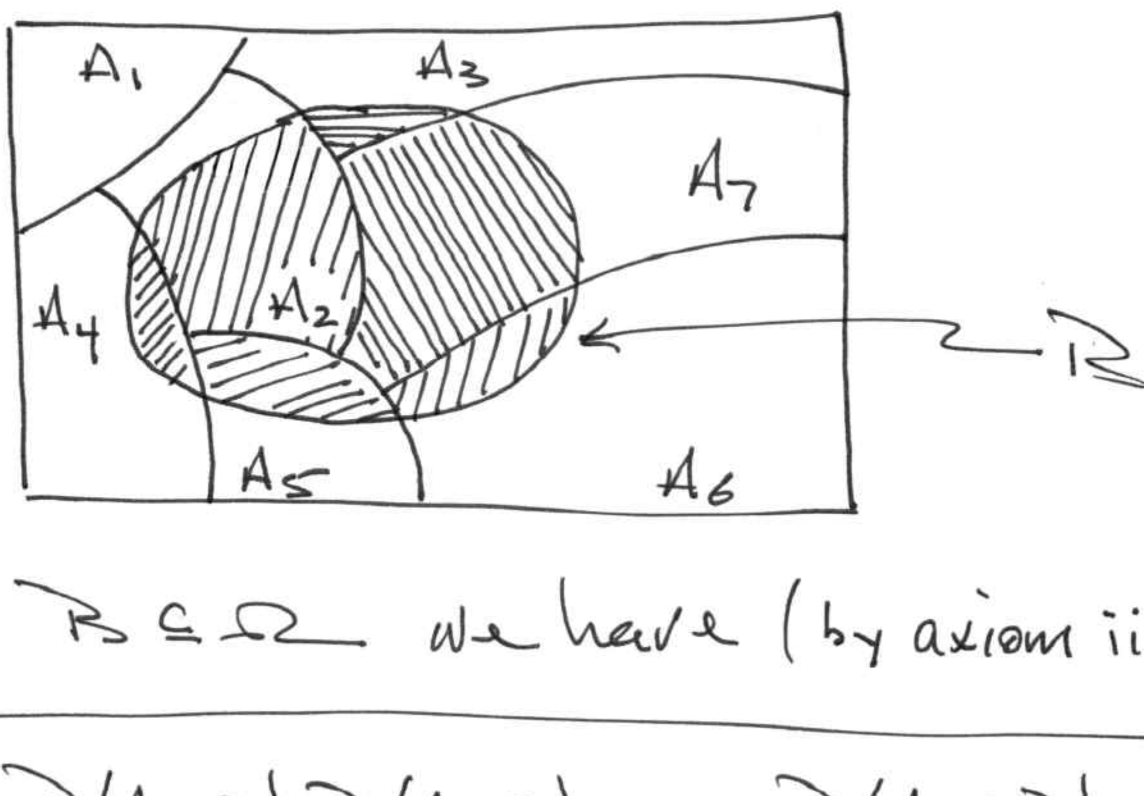
TT $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

sum = 1

1.4 Total Probability Theorem & Bayes Rule

Let $A_1, A_2, \ldots, A_n$ be a Partition of Ω, i.e. pairwise disjoint and union is Ω.

Ω



For any $B \subseteq \Omega$ we have (by axiom iii)

$$\boxed{P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)}$$

If $P(A_i) > 0$ for all $i = 1, \ldots, n$, then

$$\boxed{P(B) = P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + \cdots + P(A_n)\cdot P(B|A_n)}$$

total probability theorem

Ex. Roll a fair die. if result is $\in \{1, 2, 3\}$, then roll again, otherwise stop. Compute sum of all rolls. What is probability of

$$B = \{\text{sum} \geq 5\}$$

Let $A_i = \{1^{st} \text{ roll is } i\}$ $(1 \le i \le 6)$.

Then $P(A_i) = \frac{1}{6}$ $(1 \le i \le 6)$

Note

if A_1 occurs, then B occurs iff 2^{nd} roll $\in \{4,5,6\}$

" A_2 " " " " " " " $\in \{3,4,5,6\}$

" A_3 " " " " " " " $\in \{2,3,4,5,6\}$

if A_4 occurs, then B does not occur

if A_5 occurs, then B does occur

" A_6 " " " " "

so

$P(B|A_1) = \frac{3}{6}$ $\quad P(B|A_4) = 0$

$P(B|A_2) = \frac{4}{6}$ $\quad P(B|A_5) = 1$

$P(B|A_3) = \frac{5}{6}$ $\quad P(B|A_5) = 1$

Thus

$$P(B) = \frac{1}{6}\cdot\frac{3}{6} + \frac{1}{6}\cdot\frac{4}{6} + \frac{1}{6}\cdot\frac{5}{6} + \cancel{\frac{1}{6}\cdot 0} + \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 1$$

$$= \frac{1}{6}\left(\frac{3}{6} + \frac{4}{6} + \frac{5}{6} + 1 + 1\right)$$

$$= \frac{1}{6}\cdot\frac{24}{6} = \frac{1}{6}\cdot 4 = \boxed{\frac{2}{3}}$$

Bayes' Rule

Let $A_1, \ldots, A_n$ form a partition of Ω. Assume $P(A_i) > 0$ for $i = 1, \ldots, n$. Let $B \subseteq \Omega$ with $P(B) > 0$.

Recall

$$P(A \cap B) = P(B) \cdot P(A|B), \quad \text{and}$$

$$P(A \cap B) = P(A) \cdot P(B|A)$$

$$\therefore \quad P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$$

$$\therefore \quad \boxed{P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}}$$

apply to A_i

$$\boxed{P(A_i|B) = \frac{P(A_i) \cdot P(B|A_i)}{P(B)}}$$

also

$$\boxed{P(A_i|B) = \frac{P(A_i)\cdot P(B|A_i)}{P(A_1\cap B)+P(A_2\cap B)+\cdots+P(A_n\cap B)}}$$

also

$$\boxed{P(A_i|B) = \frac{P(A_i)\cdot P(B|A_i)}{P(A_1)\cdot P(B|A_1)+\cdots\cdots+P(A_n)\cdot P(B|A_n)}}$$

Baye's Rule

$$\begin{array}{ccc} A_1 & \rightarrow & \\ A_2 & \rightarrow & B \\ \vdots & \rightarrow & \\ A_n & & \\ \text{causes} & & \text{effect} \end{array}$$

$P(B|A_i)$ is the measure strength of causation

Causation: $A_i \rightarrow B : P(B|A_i)$

Given effect B, we wish to evaluate $P(A_i|B)$

Inference: $A_i \leftarrow B : P(A_i|B)$

$$\underset{\text{Posterior}}{P(A_i|B)} = \frac{\overset{\text{Prior}}{P(A_i)} \cdot \overset{\text{likelihood}}{P(B|A_i)}}{\underset{\text{marginal}}{P(B)}}$$

Ex. Disease Problem again

$T = \{$ ~~test~~ positive $\}$

$D = \{$ has disease $\}$

$P(T|D) = .99$

$P(T|D^c) = .10$

$P(D) = .05, \quad P(D^c) = .95$

Find $P(D|T)$. using Baye's Rule

$$P(D|T) = \frac{P(D) \cdot P(T|D)}{P(T)}$$

$$= \frac{P(D) \cdot P(T|D)}{P(D) \cdot P(T|D) + P(D^c) \cdot P(T|D^c)}$$

$$= \frac{(.05)(.99)}{(.05)(.99) + (.95)(.10)}$$

$$= .34256 \approx \boxed{.3426}$$

round to 4 dec. places