cse 107 4-3-25

continue

$P(\text{sum}=4) = 3/36$

$P(\text{"} = 5) = 4/36$

$P(\text{"} = 6) = 5/36$

$P(\text{"} = 7) = 6/36 \leftarrow$ max

$P(\text{"} = 8) = 5/36$

$P(\text{"} = 9) = 4/36$

$P(\text{"} = 10) = 3/36$

$P(\text{"} = 11) = 2/36$

$P(\text{"} = 12) = 1/36$

Exercise

Determine

P(sum is even)

P(" " odd)

P(max = 4)

P(min = 4)

P(1st roll > 2nd roll)

P(1st = 2nd roll) = 6/36

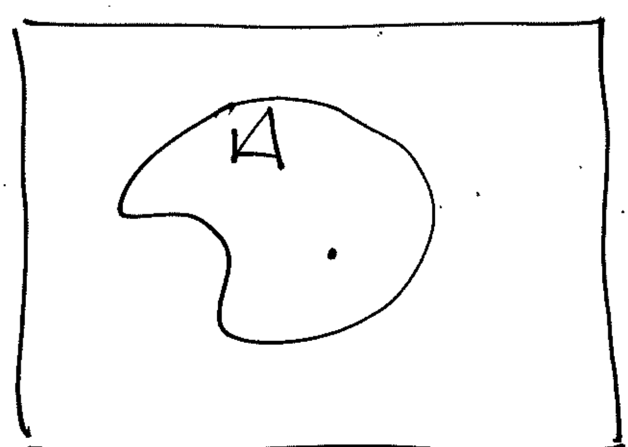
P(at least one roll is 3)

Continuous Models

If Ω is a continuum of some kind (interval, region, volume in $\mathbb{R}^3$), then the probabilities $P(\omega)$ for each $\omega \in \Omega$ are not sufficient to find $P(A)$ for $A \subseteq \Omega$ ... often:

constant

$$P(A) = c \cdot \text{area}(A)$$



Ω

uniform law

Ex.

R and J have a date, and each will be delayed by a random time in range 0 to 1 hour.

The 1^{st} to arrive will wait 15 minutes, then leave.

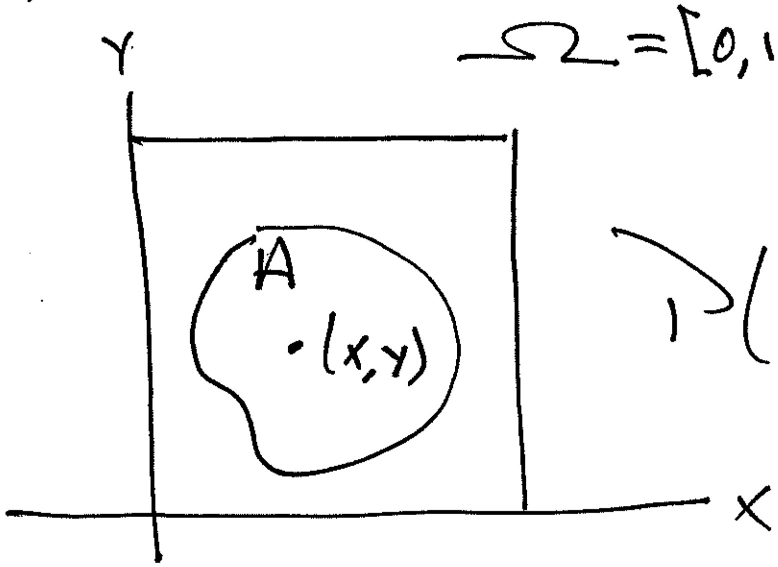
What is the probability that they meet?

let x be R's delay time, and y J's delay. Thus

$$0 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1$$

The word random means all all delay pairs (x, y) are equally likely.

$\Omega = [0,1] \times [0,1]$



$P(A) = \text{const} \cdot \text{area}(A)$

const=1.

eq. likely means uniform, so

$\therefore\ P(A) = \text{area}(A)$ in this case.

<u>note</u>: $P(\{(x,y)\}) = 0$

We seek $P(M)$ where M is the region

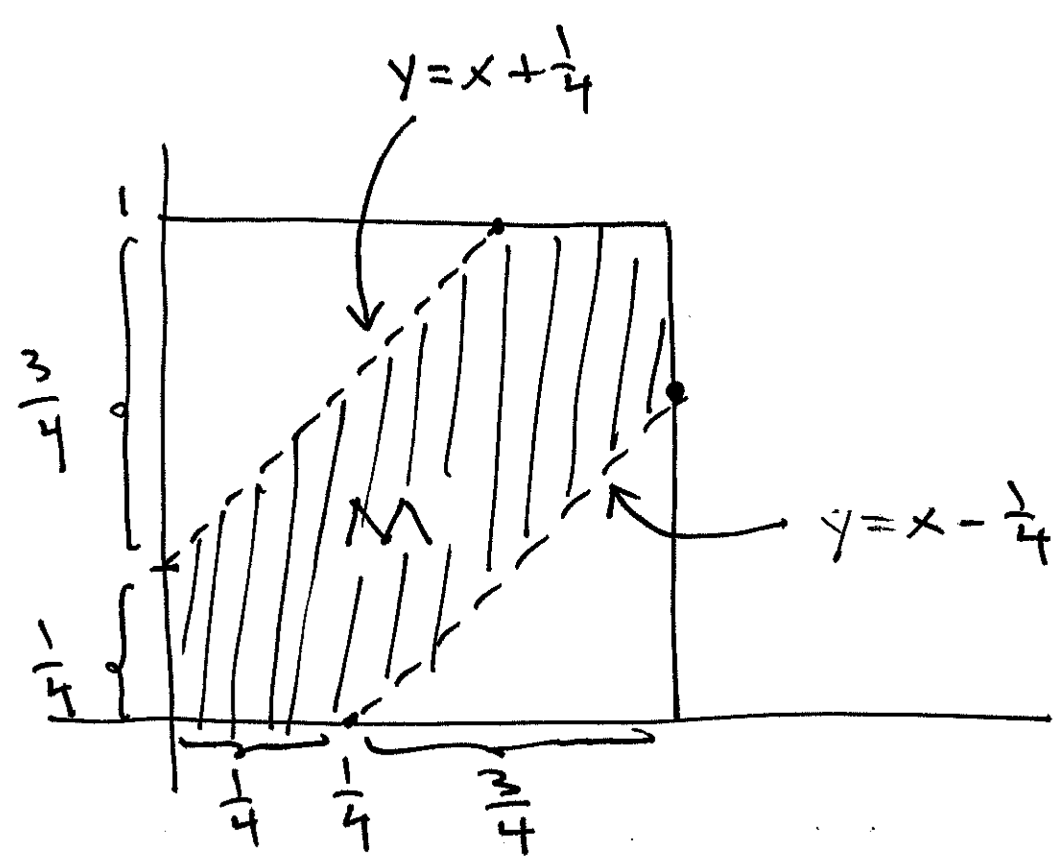
$$M = \{(x,y) \mid \underbrace{|x-y| \le \tfrac{1}{4}},\ (x,y) \in \Omega\}$$

$$|x-y| \le \tfrac{1}{4}$$

$$\therefore\ -\tfrac{1}{4} \le x - y \le \tfrac{1}{4}$$

$$\swarrow \qquad\qquad \searrow$$

$y \le x + \frac{1}{4}$ <u>and</u> $y \ge x - \frac{1}{4}$



so

$$P(M) = \text{area}(M) = 1 - \left(\tfrac{3}{4}\right)^2$$

$$= 1 - \frac{9}{16} = \boxed{\frac{7}{16}}$$

In general, if a continuous <u>uniform</u> model has $\Omega \subseteq \mathbb{R}^2$, then for $A \subseteq \Omega$.

$$P(A) = \text{const} \cdot \text{area}(A)$$

since $P(\Omega) = 1$, we have

$$1 = P(\Omega) = \text{const} \cdot \text{area}(\Omega)$$

$$\therefore \quad \text{const} = \frac{1}{\text{area}(\Omega)} .$$

so

$$P(A) = \frac{\text{area}(A)}{\text{area}(\Omega)}$$

Exercise

Alice & Bob each pick a random number in $[0, 2]$. let Alice's # be x, and Bob's # be y.

what is the Probability that

$$2x > y^2 \text{ and } 2y > x^2$$

Properties of $P(\cdot)$

more Properties:

✓(a) if $A \subseteq B$, then $P(A) \leq P(B)$

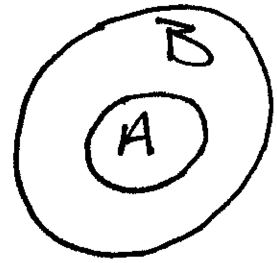
✓(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

✓(c) $P(A \cup B) \leq P(A) + P(B)$

✓(d) $P(A \cup B \cup C) = P(A) + P(A^c \cap B)$
$+ P(A^c \cap B^c \cap C)$

(a) Proof

$$A \subseteq B \Rightarrow \begin{cases} B = A \cup (B-A) \\ A \cap (B-A) = \phi \end{cases}$$

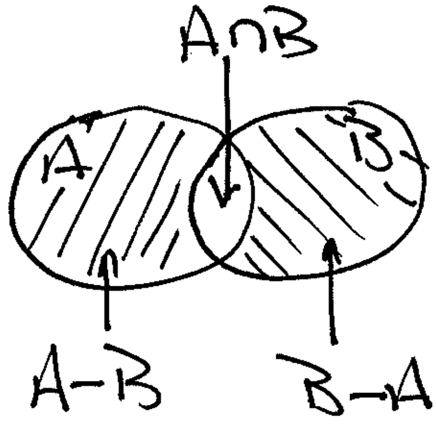


$\therefore P(B) = P(A) + P(B-A) \geq P(A)$

$\therefore \quad P(A) \leq P(B)$ ∎

(b) Proof

$A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$



disjoint union

$\therefore P(A \cup B) = P(A-B) + P(A \cap B) + P(B-A)$

also note:

$$B = (B-A) \cup (A \cap B) \quad \Big\{ \begin{matrix}\text{disjoint} \\ \text{union}\end{matrix}$$

$$\therefore P(B) = P(B-A) + P(A \cap B)$$

$$\therefore \bullet\, P(B-A) = P(B) - P(A \cap B)$$

similarly

$$\bullet\, P(A-B) = P(A) - P(A \cap B)$$

Thus

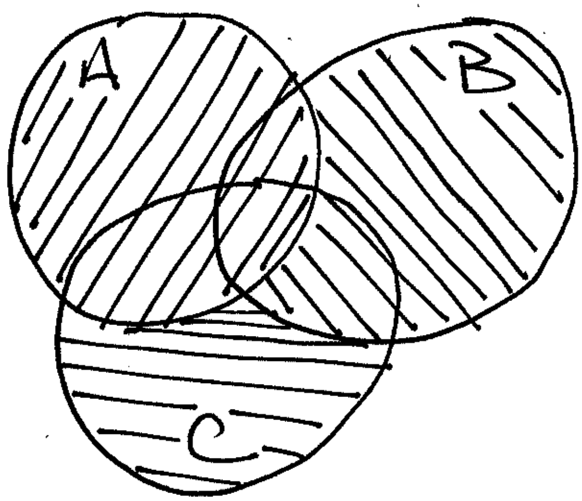
$$P(A \cup B) = \big(P(A) - \cancel{P(A \cap B)}\big) + \cancel{P(A \cap B)}$$
$$+ \big(P(B) - P(A \cap B)\big)$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

$\blacksquare$

(c) follows from (b).

(d) Proof



sets: A, $A^c \cap B$, $A^c \cap B^c \cap C$

are pairwise disjoint & have union

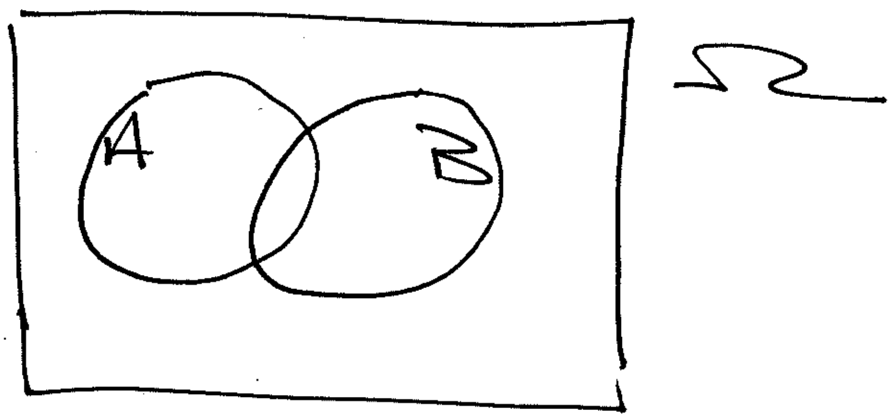
$A \cup B \cup C$. By axiom 3:

$$P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C).$$

$\blacksquare$

1.3 Conditional Probability

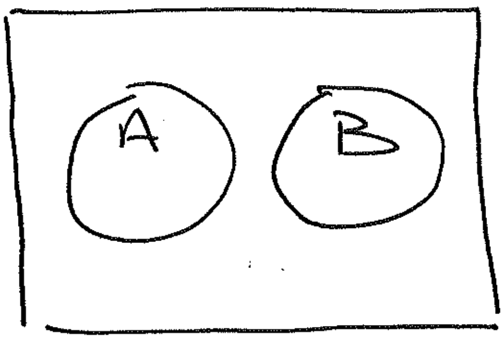
Suppose A, B are events in Ω.



How does Prob. of A change with the knowledge that B occurs.

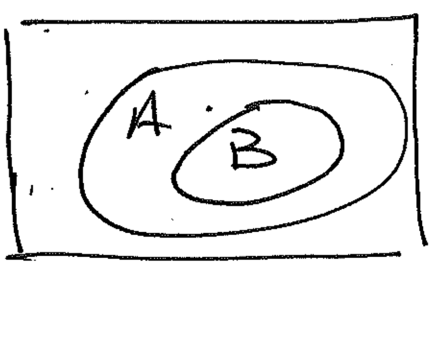
Two extreme cases:

- $A \cap B = \phi \Rightarrow \begin{cases} \boxed{A \subseteq B^c} \\ \boxed{B \subseteq A^c} \end{cases} \leftarrow$

Ω 

$\therefore$ B occurs $\Rightarrow$ A <u>does</u> not occur

- $B \subseteq A$

Ω 

$\therefore$ B occurs $\Rightarrow$ A <u>does</u> occur

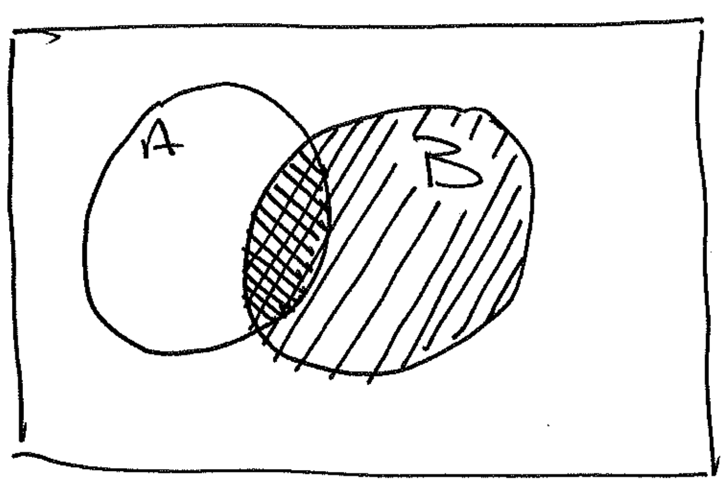
Defn

The conditional Probability of A given B, denoted By $P(A|B)$ is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

We assume $P(B) > 0$.

Picture:



Ω

observe conditional Probability given B

$$P(\cdot \mid B)$$

is itself a valid Probability law, since it satisfies Axioms.

(i) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} \geq 0$

(ii) $P(\Omega|B) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

(iii) 2 set version:

If $A_1 \cap A_2 = \phi$, then

$$P(A_1 \cup A_2 | B) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)}$$

$$= \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1 | B) + P(A_2 | B). \blacksquare$$