CSE 107 4-24-25

Defn

Two r.v.s X, Y are conditionally indep. given $A \subseteq \Omega$ iff

$$P(X=x, Y=y \mid A) = P(X=x \mid A) \cdot P(Y=y \mid A)$$

i.e.

$$p_{X,Y|A}(x,y) = p_{X|A}(x) \cdot p_{Y|A}(y)$$

note indep. $\not\Leftrightarrow$ cond. indep.

See p. 111 of text.

Consequence of independence

- $$\boxed{E[X\cdot Y] = E[X]\cdot E[Y]}$$

Proof

Suppose X, Y are independent.

$$E[X\cdot Y] = \sum_x \sum_y x\cdot y \, p_{X,Y}(x,y)$$

$$= \sum_x \sum_y x y \, p_X(x) p_Y(y)$$

$$= \sum_x x \, p_X(x) \cdot \left(\sum_y y \, p_Y(y)\right)$$

$$= \left(\sum_x x \, p_X(x)\right) \cdot \left(\sum_y y \, p_Y(y)\right)$$

$$= E[X] \cdot E[Y].$$

$\blacksquare$

Thm

If X, Y are indep., then so are $g(X), h(Y)$ for any $g, h : \mathbb{R} \to \mathbb{R}$.

see Problem #44 on p. 134 of text.

Thus

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)].$$

Another consequence is that

$$Var(X+Y) = Var(X) + Var(Y)$$

Proof

let X, Y be indep. let

$$U = X - E[X]$$
$$V = Y - E[Y]$$

so

$$E[U] = E[X] - E[X] = 0$$

and $E[V] = - \quad - \quad - \quad - = 0$

Also

$$\mathrm{Var}(U) = \mathrm{Var}(X)$$

and $\mathrm{Var}(V) = \mathrm{Var}(Y)$

Also

$$U+V = X - E[X] + Y - E[Y]$$
$$= (X+Y) - E[X+Y].$$

Thus $E[U+V] = 0$, $\mathrm{Var}(U+V) = \mathrm{Var}(X+Y)$.

so

$$\mathrm{Var}(X+Y) = E\left[\left((X+Y) - E[X+Y]\right)^2\right]$$
$$= E\left[(U+V)^2\right]$$

$$= E[U^2 + 2UV + V^2]$$

$$= E[U^2] + 2E[UV] + E[V^2]$$

$$= E[U^2] + \underbrace{2E[U]\cdot E[V]}_{\text{by indep.}} + E[V^2]$$

$$= E[(X - E[X])^2] + \cancel{2\cdot 0\cdot 0}$$

$$+ E[(Y - E[Y])^2]$$

$$= Var(X) + Var(Y).$$

$\blacksquare$