cse 107 4-22-25

- Midterm 1 : Th 4/24/25

  5:20 – 6:25

<u>Ex.</u> (the Professor Problem).

A professor answers questions wrongly with probability $\frac{1}{4}$, independent of other questions. The # of questions asked is <u>0</u>, <u>1</u> or 2 with equal probability $\frac{1}{3}$. let

X = # questions asked.

Y = # questions answered <u>wrongly</u>.

find joint PMF of X, Y

We have

$$P_X(x) = \begin{cases} \frac{1}{3} & \text{if } x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

we also have

$$P(Y=0 \mid X=0) = 1$$

$$P(Y=0 \mid X=1) = \frac{3}{4}$$

$$P(Y=1 \mid X=1) = \frac{1}{4}$$

$$P(Y=0 \mid X=2) = \frac{9}{16}$$

$$P(Y=1 \mid X=2) = \frac{3}{16} + \frac{3}{16} = \frac{6}{16} = \frac{3}{8}$$

$$P(Y=2 \mid X=2) = \frac{1}{16}$$

We have:

$P_X(x)$

x	0	1	2
$P_X(x)$	1/3	1/3	1/3

$P_{Y|X}(y|x)$

$y \backslash x$	0	1	2
2	0	0	1/16
1	0	1/4	3/8
0	1	3/4	9/16

Want $P_{X,Y}(x,y) = P_{Y|X}(y|x) \cdot P_X(x)$

$P_{X,Y}(x,y)$

$y \backslash x$	0	1	2		$P_Y(y)$
2			1/48	$\longrightarrow$	1/48
1		1/12	1/8	$\longrightarrow$	10/48
0	1/3	1/4	3/16	$\longrightarrow$	37/48

$$P_Y(y) = \sum_x P_{X,Y}(x,y)$$

Also $p_{X|Y}(x|y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)}$

$p_{X|Y}(x|y)$

$$
\begin{array}{c|ccc}
y & & & \\
2 & & & 1 \\
1 & & 2/5 & 3/5 \\
0 & 16/37 & 12/37 & 9/37 \\
\hline
 & 0 & 1 & 2 \\
 & \multicolumn{3}{c}{\underbrace{\hphantom{0 \quad 1 \quad 2}}_{x}}
\end{array}
$$

Conditional expectation

Let X be a r.v., $A \subseteq \Omega$, $\mathbb{P}(A) > 0$.

Define

$$\boxed{\mathbb{E}[X|A] = \sum_x x \cdot \mathbb{P}_{X|A}(x)}$$

expected value rule

$$\boxed{\mathbb{E}[g(X)|A] = \sum_x g(x) \cdot \mathbb{P}_{X|A}(x)}$$

We can condition on $A = \{Y = y\}$

provided $\mathbb{P}(Y = y) > 0$.

$$\boxed{E[X|Y=y] = \sum_x x\, p_{X|Y}(x|y)}$$

Also, if $A_1, A_2, \ldots, A_n$ form a partition of Ω, then

Total Expectation Theorem:

$$\boxed{E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X|A_i]}$$

Proof

by the total probability theorem

$$p_X(x) = \sum_{i=1}^{n} P(A_i) \cdot p_{X|A_i}(x)$$

i.e.

$$P(X=x) = \sum_{i=1}^{n} P(A_i) \cdot P(X=x \mid A_i)$$

multiply both sides by x, sum over x to obtain

$$E[X] = \sum_{x} x\, p_X(x)$$

$$= \sum_{x} x \cdot \sum_{i=1}^{n} P(A_i) \cdot p_{X|A_i}(x)$$

$$= \sum_{x} \sum_{i=1}^{n} P(A_i) \cdot x\, p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \sum_{x} x\, p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot E[X | A_i]$$

$\blacksquare$

- see similar identities in summary on p. 104.

Mean & Variance of Geometric r.v.

Recall Geometric(p) has PMF

$$p_X(k) = (1-p)^{k-1} \cdot p \quad (k = 1, 2, 3, \ldots)$$

Then

$$E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p ,$$

$$E[X^2] = \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p$$

and

$$Var(X) = E[X^2] - E[X]^2$$

We use total expectation Thm.

Let

$$A_1 = \{X=1\} = \{1^{st} \text{ flip is head}\}$$

$$A_2 = \{X>1\} = \{1^{st} \text{ flip is tail}\}$$

Then $P(A_1) = p$, $P(A_2) = 1-p$.

Note: $E[X|A_1] = 1$, and

$$E[X|A_2] = E[1 + X] = 1 + E[X]$$

Thus

$$E[X] = P(A_1) \cdot E[X|A_1] + P(A_2) \cdot E[X|A_2]$$

$$= p \cdot 1 + (1-p)(1 + E[X])$$

$$\cancel{E[X]} = \cancel{p} + 1 + \cancel{E[X]} - \cancel{p} - p \cdot E[X]$$

$$\therefore \quad p\,E[X] = 1$$

$$\therefore \quad \boxed{E[X] = \frac{1}{p}}.$$

Similarly

$$E[X^2 \mid A_1] = 1$$

and

$$E[X^2 \mid A_2] = E[(1+X)^2]$$

$$= E[1 + 2X + X^2]$$

$$= 1 + 2E[X] + E[X^2]$$

so

$$E[\cancel{X}^2] = P(A_1) \cdot E[X^2 \mid A_1] + P(A_2) E[X^2 \mid A_2]$$

$$= p \cdot 1 + (1-p)(1 + 2E[X] + E[X^2])$$

$$= p + (1-p)\left(1 + \frac{2}{p} + E[X^2]\right)$$

$$= \cancel{p} + (1-\cancel{p}) + 2\left(\frac{1-p}{p}\right) + (\cancel{1}-p) E[X^2]$$

$$\therefore\ p\,E[X^2] = 1 + \frac{2}{p} - 2 = \frac{2}{p} - 1$$

$$\therefore\ E[X^2] = \frac{2}{p^2} - \frac{1}{p}$$

finally

$$Var(X) = \left(\frac{2}{p^2} - \frac{1}{p}\right) - \left(\frac{1}{p}\right)^2$$

$$= \frac{1}{p^2} - \frac{1}{p}$$

$$= \boxed{\frac{1-p}{p^2}}$$

2.7 Independence

Defn

we say two r.v.s X, Y are independent iff for all $x \in \text{range}(X)$, $y \in \text{range}(Y)$ the events

$$\{X = x\}, \{Y = y\}$$

are independent, i.e.

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

equivalently

$$* \quad \boxed{P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y)}$$

if $P_Y(y) > 0$, this is equiv. to

$$\frac{P_{X,Y}(x,y)}{P_Y(y)} = P_X(x)$$

i.e.

$$\boxed{P_{X|Y}(x|y) = P_X(x)}$$