cse 107 4-18-25

Supplemental Lecture

Ex. (previous)

$$E[X+3Y] = E[X] + 3E[Y]$$

$$= \left(1\cdot\frac{4}{20} + 2\cdot\frac{5}{20} + 3\cdot\frac{6}{20} + 4\cdot\frac{5}{20}\right)$$

$$+3\left(1\cdot\frac{6}{20} + 2\cdot\frac{9}{20} + 3\cdot\frac{5}{20}\right)$$

$$= \boxed{\frac{169}{20}}$$

We can do same stuff for 3, 4, 5, 6, ... r.v.s .

Suppose X, Y, Z are r.v.s, then their joint PMF is

$$p_{X,Y,Z}(x,y,z) = P(X=x, Y=y, Z=z)$$

marginal PMFs:

$$p_{X,Y}(x,y) = \sum_z p_{X,Y,Z}(x,y,z)$$

$$p_{X,Z}(x,z) = \sum_y p_{X,Y,Z}(x,y,z)$$

$$p_{Y,Z}(y,z) = \sum_x p_{X,Y,Z}(x,y,z)$$

Expected value rule

$$\boxed{E[g(X,Y,Z)] = \sum_{x,y,z} g(x,y,z)\, p_{X,Y,Z}(x,y,z)}$$

let

$$X_1, X_2, \ldots\ldots, X_n$$

be r.v.s. Then

$$E[a_1 X_1 + a_2 X_2 + \cdots + a_n X_n + b]$$

$$= a_1 E[X_1] + \cdots + a_n E[X_n] + b$$

The mean of the Binomial

let X be Binomial(n,p). Recall

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad (k = 0, 1, \ldots, n)$$

so

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

<u>note</u>: $X = X_1 + X_2 + \cdots + X_n$

where each X_i is Bernoulli(p)

Recall: $E[X_i] = 1\cdot p + 0\cdot(1-p) = p$, so

$$E[X] = E[X_1 + \cdots + X_n]$$
$$= E[X_1] + E[X_2] + \cdots + E[X_n]$$
$$= p + p + \cdots + p$$
$$= np$$

Also

$$Var(X) = np(1-p)$$

Proof later. need this 2.4 #22 on P.123 (discussion problem.)

Ex The hat Problem

Suppose n people throw their hats into a box, then each picks one hat (at random) from box so each bijection

$$\{\text{hats}\} \longrightarrow \{\text{people}\}$$

is equally likely. Let $X = \#$ of people who get their own hat back. Find $E[X]$.

observe

$$X = X_1 + X_2 + \cdots + X_n$$

where each X_i is Bernoulli$\left(\frac{1}{n}\right)$

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ person own hat} \\ 0 & \text{otherwise} \end{cases}$$

we have

$$
\begin{aligned}
E[X] &= E[X_1 + \cdots + X_n] \\
&= E[X_1] + \cdots + E[X_n] \\
&= \frac{1}{n} + \cdots + \frac{1}{n} \\
&= n \cdot \frac{1}{n} = \boxed{1}
\end{aligned}
$$

<u>note</u> X is not Binomial, since the Bernoulli trials are not independent.

2.6 Conditioning

Given a r.v. $X : \Omega \to \mathbb{R}$.

any valid probability law can be used to define PMF of X.

$$p_X(x) = \mathbb{P}(X = x)$$

This includes a conditional prob. law $\mathbb{P}(\cdot \mid A)$, where $A \subseteq \Omega$ with $\mathbb{P}(A) > 0$.

Defn
Let X be a r.v., $A \subseteq \Omega$, $\mathbb{P}(A) > 0$. The conditional PMF of X given A is

$$\boxed{p_{X|A}(x) = \mathbb{P}(X = x \mid A) = \frac{\mathbb{P}(\{X = x\} \cap A)}{\mathbb{P}(A)}}$$

note

$$A = \bigcup_{x \in \text{range}(X)} \left(\{X = x\} \cap A \right)$$

is a disjoint union, so by total probability we have

$$P(A) = \sum_x P(\{X = x\} \cap A).$$

Thus

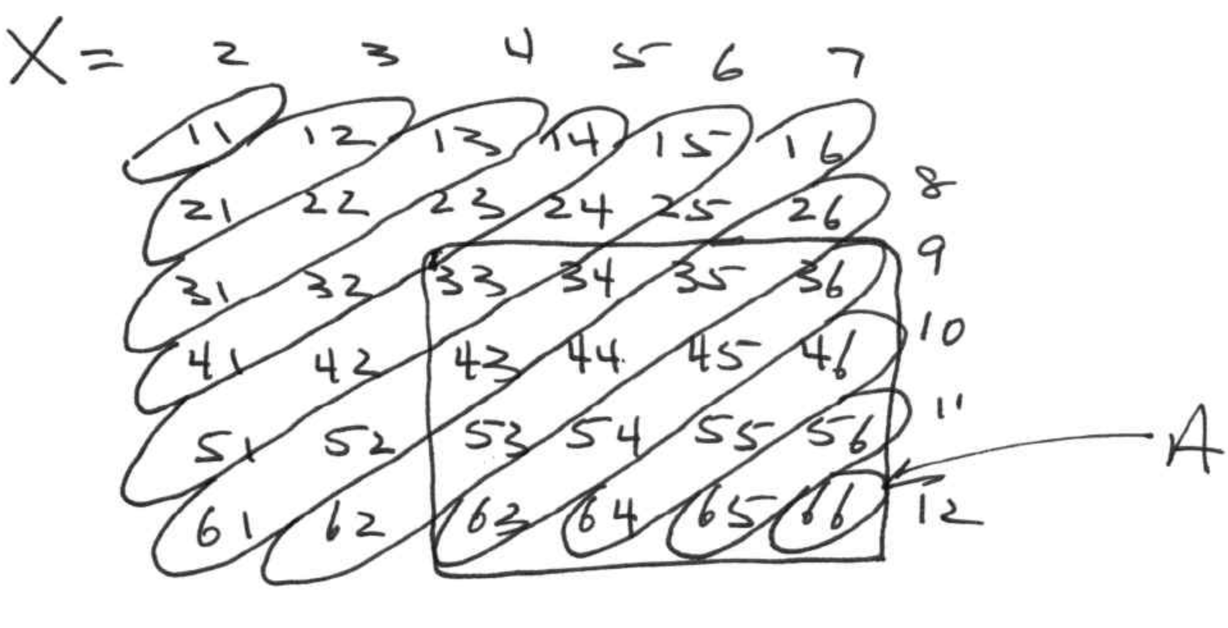
$$\sum_x P_{X|A}(x) = \sum_x \frac{P(\{X = x\} \cap A)}{P(A)}$$

$$= \frac{P(A)}{P(A)} = 1$$

so $P_{X|A}(x)$ is a valid PMF.

Ex. roll 2 indep. fair 6-sided dice, and let X be sum of top faces.

$$p_X(k) = \begin{cases} 1/36 & \text{if } k = 2, 12 \\ 2/36 & \text{if } k = 3, 11 \\ 3/36 & \text{if } k = 4, 10 \\ 4/36 & \text{if } k = 5, 9 \\ 5/36 & \text{if } k = 6, 8 \\ 6/36 & \text{if } k = 7 \\ 0 & \text{otherwise} \end{cases}$$



let $A = \{\min \geq 3\}$, so $P(A) = \frac{16}{36}$.

Then

$$P_{X|A}(7) = \frac{P(\{X=7\} \cap A)}{P(A)} = \frac{2/36}{16/36} = \frac{2}{16}.$$

similarly

$$P_{X|A}(k) = \begin{cases} 1/16 & \text{if } k = 6, 12 \\ 2/16 & \text{if } k = 7, 11 \\ 3/16 & \text{if } k = 8, 10 \\ 4/16 & \text{if } k = 9 \\ 0 & \text{otherwise} \end{cases}$$

Ex. flip a weighted coin with

$$P(H) = p$$

until the first head. let X be # flips. Then

$$P_X(k) = \begin{cases} (1-p)^{k-1} p & \text{if } k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Let $A = \{1^{st}$ head within n flips$\} = \{X \leq n\}$

Then

$$P(A) = \sum_{k=1}^{n} P_X(k) = \sum_{k=1}^{n} (1-p)^{k-1} \cdot p \quad \{k \leftarrow k+1$$

$$= p \cdot \sum_{k=0}^{n-1} (1-p)^k$$

$$= p \cdot \frac{1-(1-p)^n}{1-(1-p)} = \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{p}}$$

$$= \boxed{1-(1-p)^n}$$

another way

$$P(A) = 1 - P(A^c)$$
$$= 1 - P(1^{st}\ n \text{ flips tails})$$
$$= \boxed{1-(1-p)^n}$$

Thus

$$P_{X|A}(k) = \frac{P(\{X=k\}\cap\overbrace{\{X\le n\}}^{A})}{P(A)} = \frac{P(X=k)}{P(A)} \quad \begin{pmatrix}\text{if}\\ k\le n\end{pmatrix}$$

$$= \frac{(1-p)^{k-1}\cdot p}{1-(1-p)^n} \quad \text{if} \quad k = 1, 2, \dots, n$$

i.e.

$$P_{X|A}(k) = \begin{cases} \dfrac{(1-p)^{k-1}\cdot p}{1-(1-p)^n} & \text{if } k = 1, 2, \dots n \\ 0 & \text{otherwise} \end{cases}$$

Let Y be another r.v.. We can specialize A to be of the form

$$A = \{Y = y\}$$

where $y \in range(Y)$, so $P(A) > 0$.

<u>Defn</u>

The conditional PMF of X given Y is

$$P_{X|Y}(x|y) = P(X = x \mid Y = y)$$

$$= \frac{P(X = x, Y = y)}{P(Y = y)}$$

equivalently

$$\boxed{P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}} \tag{1}$$

also

$$\boxed{P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}} \tag{2}$$

$$(1) \Longrightarrow \boxed{P_{X,Y}(x,y) = P_{X|Y}(x|y) \cdot P_Y(y)}$$

$$(2) \Longrightarrow \boxed{P_{X,Y}(x,y) = P_{Y|X}(y|x) \cdot P_X(y)}$$