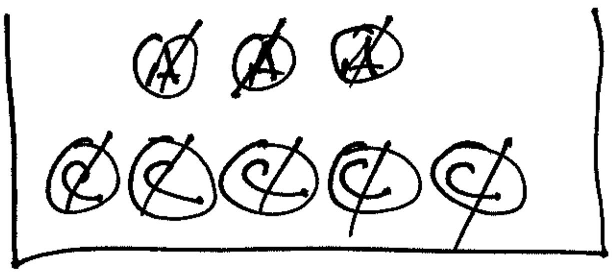
cse 107 4-17-25

Rmks on lab 2:

let a = 3, c = 5



1 start: C C A
replace

2 restart: C A
replace

3 restart: A A C
replace

4 restart: A C replace

5 restart: C C ← last ball is Carmine

continue Discrete Uniform r.v.

To compute $Var(X)$, first set

$Y = X + (1-a)$, so

$$Var(Y) = Var\left(X + (1-a)\right) = Var(X)$$

let $n = b - a + 1$, note Y is discrete uniform r.v. on $\{1, 2, \ldots, n\}$

since

$$X = a \rightarrow Y = a + (1-a) = 1$$

$$X = b \rightarrow Y = b + (1-a) = n$$

Thus

$$E[Y] = \frac{n+1}{2}$$

and

$$E[Y^2] = \sum_{k=1}^{n} k^2 \cdot \frac{1}{n}$$

$$= \frac{1}{n} \cdot \sum_{k=1}^{n} k^2$$

$$= \frac{1}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

Thus

$$Var(X) = Var(Y) = E[Y^2] - E[Y]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \text{.....exercise.....}$$

$$= \frac{(n+1)(n-1)}{12} = \frac{(b-a)(b-a+2)}{12}$$

Ex. The Poisson(λ)

$$P_X(k) = e^{-\lambda}\cdot\frac{\lambda^k}{k!} \quad (k=0,1,2,\dots)$$

We show that

(1) $E[X] = \lambda$

(2) $E[X^2] = \lambda^2 + \lambda$

Proof of (1)

$$E[X] = \sum_{k=\cancel{0}\,1}^{\infty} k\cdot e^{-\lambda}\cdot\frac{\lambda^k}{k!}$$

$$= e^{-\lambda}\cdot\sum_{k=1}^{\infty} k\cdot\frac{\lambda^k}{k!} \qquad (*)$$

$$= \lambda e^{-\lambda}\cdot\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} \qquad \left\{\begin{array}{l}\text{replace}\\ k \leftarrow k+1\end{array}\right.$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= \lambda \cdot \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}} = \boxed{\lambda}$$

<u>Proof of (2)</u>

$$E[X^2] = \sum_{k=\cancel{0}1}^{\infty} k^2 \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^{k-1}}{(k-1)!} \quad \left\{ \begin{array}{l} \text{replace} \\ k \leftarrow k+1 \end{array} \right.$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=0}^{\infty} (k+1) \cdot \frac{\lambda^k}{k!}$$

$$= \lambda e^{-\lambda} \left(\sum_{k=\cancel{0}1}^{\infty} k \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right)$$

$$= \lambda \left(e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} + e^{-\lambda} \cdot e^{\lambda} \right)$$

$$= \lambda(\lambda + 1)$$

$$= \lambda^2 + \lambda$$

Thus

$$Var(X) = E[X^2] - E[X]^2$$

$$= (\cancel{\lambda^2} + \lambda) - \cancel{\lambda^2}$$

$$= \boxed{\lambda}$$

Ex. Quiz Problem

A game consists of 2 questions

	Q_1	Q_2
P(correct)	p_1	p_2
payoff \$	v_1	v_2

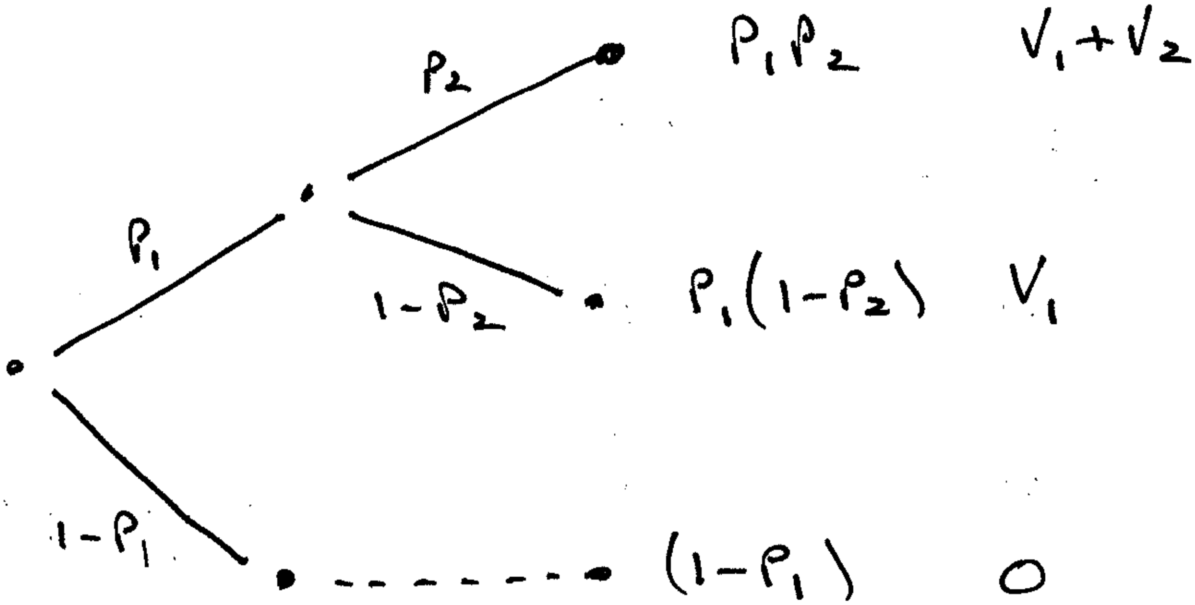
The 1^{st} wrong answer terminates the game. Which order do you want: (Q_1, Q_2), (Q_2, Q_1)

Let X = total payoff.

find $E[X]$

1st find PMF of X.

Assume order (Q_1, Q_2)



$$P_X(x) = \begin{cases} p_1 p_2 & \text{if } x = v_1 + v_2 \\ p_1(1-p_2) & \text{if } x = v_1 \\ (1-p_1) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

so

$$E[X] = (v_1+v_2)p_1p_2 + v_1p_1(1-p_2) + 0$$

$$= v_1p_1 + v_2p_1p_2$$

let Y by payoff for order (Q_2, Q_1)

obviously

$$E[Y] = v_2p_2 + v_1p_1p_2$$

should choose (Q_1, Q_2) iff

$$E[X] > E[Y]$$

i.e. $v_1 p_1 + v_2 p_1 p_2 > v_2 p_2 + v_1 p_1 p_2$

$\therefore \quad v_1 p_1 - v_1 p_1 p_2 > v_2 p_2 - v_2 p_1 p_2$

$\therefore \quad v_1 p_1 (1 - p_2) > v_2 p_2 (1 - p_1)$

$$\therefore \quad \frac{v_1 p_1}{1 - p_1} > \frac{v_2 p_2}{1 - p_2}$$

<u>Exercise</u>

- analyze problem for 3 questions
- analyze " " n questions,

<u>Conjecture</u>: sort questions by decreasing value of $f(p, v) = \frac{vp}{1-p}$

2.5 Joint PMFs, multiple r.v.s

Defn

Give 2 r.v.s X, Y their **joint PMF** is

$$p_{X,Y}(x,y) = \underbrace{P(X=x, Y=y)}_{\text{shorthand for}}$$

$$P(\{X=x\} \cap \{Y=y\})$$

for instance, if

$$S \subseteq \text{range}(X) \times \text{range}(Y) \subseteq \mathbb{R}^2$$

Then

$$P\big((x,y)\in S\big) = \sum_{(x,y)\in S} P_{X,Y}(x,y)$$

Ex.

suppose

$$\text{range}(X) = \{1,2,3,4\}$$

and

$$\text{range}(Y) = \{1,2,3\}$$

let the joint PMF of X, Y

be

Y \ x	1	2	3	4	$p_Y(y)$
3	1/20	1/20	2/20	1/20	5/20
2	2/20	3/20	3/20	1/20	9/20
1	1/20	1/20	1/20	3/20	6/20
$p_X(x)$	4/20	5/20	6/20	5/20	

$p_{X,Y}(x,y)$

suppose $Y = 3$.

note

$$\{X=1\} \cup \{X=2\} \cup \{X=3\} \cup \{X=4\} = \Omega$$

is a disjoint union, so

By total Prob. thm

$$P(Y=3) = P(X=1, Y=3) + P(X=2, Y=3)$$
$$+ P(X=3, Y=3) + P(X=4, Y=3)$$
$$= \sum_{x=1}^{4} P_{X,Y}(x, 3) = \frac{5}{20}$$

Thus

$$\boxed{P_Y(y) = \sum_x P_{X,Y}(x, y)}$$ marginal PMF

also

$$\boxed{P_X(x) = \sum_y P_{X,Y}(x, y)}$$ marginal PMF

Functions of multiple r.v.s

let $Z = g(X, Y)$, where

$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Then

$$P_Z(z) = \sum_{\{(x,y) \mid g(x,y) = z\}} P_{X,Y}(x,y)$$

2-variable expected value rule

$$\boxed{E[Z] = E[g(X,Y)] = \sum_x \sum_y g(x,y) \cdot P_{X,Y}(x,y)},$$

Exercise Prove this.

Ex show that

$$E[aX+bY+c] = aE[X]+bE[Y]+c$$

Proof

$$E[aX+bY+c] = \sum_x \sum_y (ax+by+c)\, p_{X,Y}(x,y)$$

$$= a\sum_x \sum_y x\, p_{X,Y}(x,y) + b\sum_x \sum_y y\, p_{X,Y}(x,y)$$

$$+ c\sum_x \sum_y p_{X,Y}(x,y)$$

$$= a\sum_x x\, p_X(x) + b\cdot \sum_y y\cdot p_Y(y) + c$$

$$= aE[X] + b\cdot E[Y] + c$$

$\blacksquare$