CSE 107 4-15-25

2.4 Expectation, Mean, Variance

Defn

The expectation (or mean) of a r.v. X is

$$E[X] = \sum_{x \in range(X)} x \cdot P_X(x)$$

Ex. (Previous)

$$E[X] = (-1)(.2) + (1)(.3) + (2)(.5) = \boxed{1.1}$$

$$E[X^2] = (1)(.5) + (4)(.5) = \boxed{2.5}$$

note: $E[X^2] \neq E[X]^2$

Expected value rule

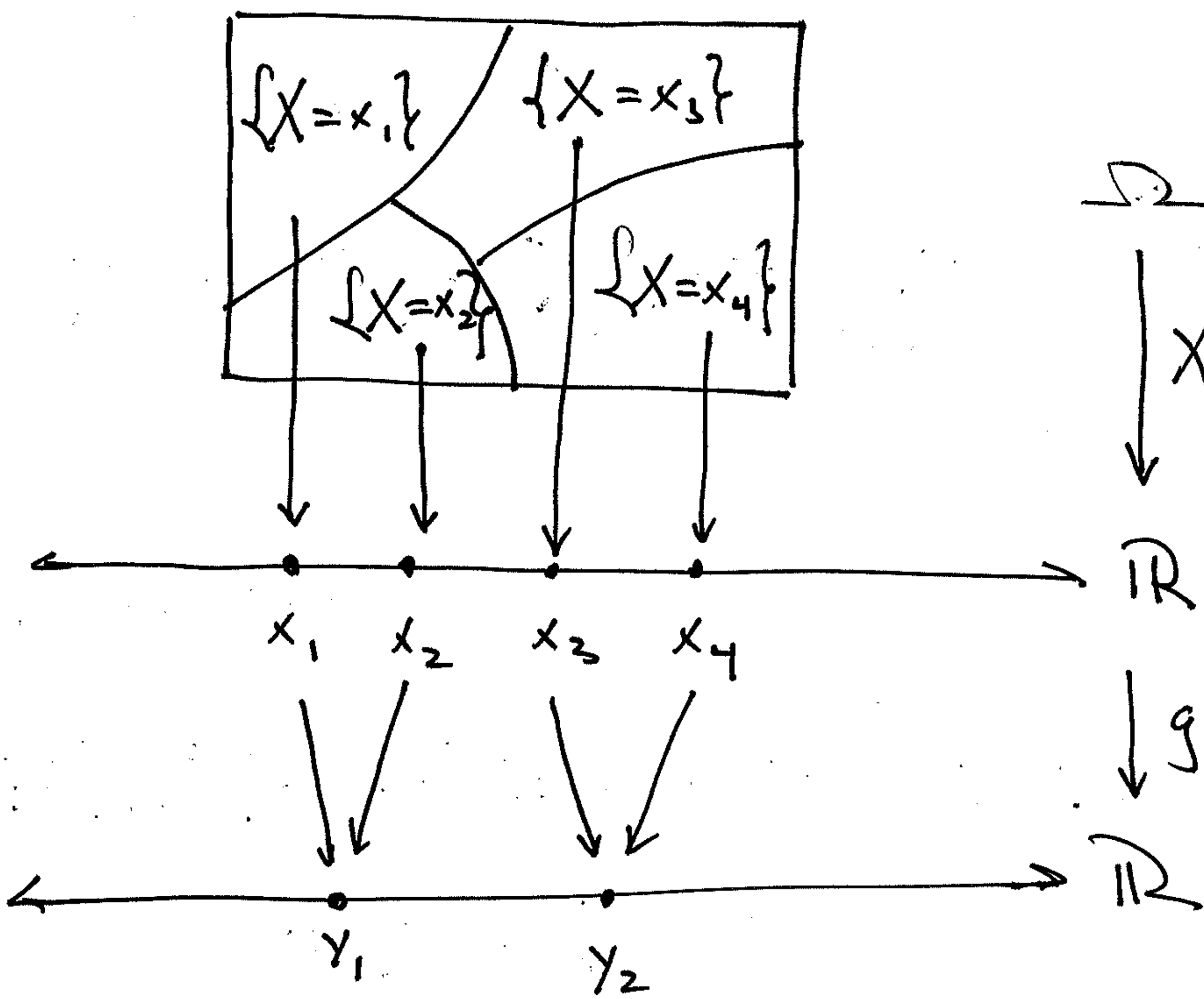
$$E[g(X)] = \sum_x g(x) \cdot p_X(x)$$

Ex. (previous)

$$E[X^2] = (-1)^2 \cdot (.2) + (1)^2 (.3) + (2)^2 (.5)$$

$$= \boxed{2.5}$$

Ex.



$$E[Y] = y_1 \cdot p_Y(y_1) + y_2 \cdot p_Y(y_2)$$

$$= y_1\left(p_X(x_1) + p_X(x_2)\right) + y_2\left(p_X(x_3) + p_X(x_4)\right)$$

$$= y_1 p_X(x_1) + y_1 p_X(x_2) + y_2 p_X(x_3) + y_2 p_X(x_4)$$

$$= g(x_1)p_X(x_1) + g(x_2)p_X(x_2) + g(x_3)p_X(x_3) + g(x_4)p_X(x_4)$$

$$= \sum_x g(x) \cdot p_X(x) .$$

Proof

Observe

$$\{Y = y\} = \{\omega \mid Y(\omega) = y\}$$

$$= \{\omega \mid g(X(\omega)) = y\}$$

$$= \bigcup_{\{x \mid g(x) = y\}} \{\omega \mid X(\omega) = x\}$$

$$= \bigcup_{\{x \mid g(x) = y\}} \{X = x\} \quad \begin{bmatrix} \text{Disjoint} \\ \text{union} \end{bmatrix}$$

$$\therefore \; P_Y(y) = P(Y=y)$$

$$= \sum_{\{x \mid g(x)=y\}} P(X=x)$$

$$= \sum_{\{x \mid g(x)=y\}} P_X(x)$$

Thus

$$E[Y] = \sum_y y \cdot P_Y(y)$$

$$= \sum_y y \cdot \sum_{\{x \mid g(x)=y\}} P_X(x)$$

$$= \sum_y \sum_{\{x \mid g(x)=y\}} y \, P_X(x)$$

$$= \sum_{y} \sum_{\{x \mid g(x) = y\}} g(x) p_X(x)$$

$$= \sum_{x} g(x) p_X(x)$$

$\blacksquare$

Ex. let $Y = aX + b \quad (a, b \in \mathbb{R})$.

Then

$$E[Y] = E[aX + b]$$

$$= \sum_{x} (ax + b) p_X(x)$$

$$= \sum_{x} \left(a x p_X(x) + b p_X(x) \right)$$

$$= a \sum_x x p_X(x) + b \sum p_X(x)$$

$$= a E[X] + b$$

so

$$\boxed{E[aX+b] = aE[X] + b}$$

In particular

$$b=0 \Rightarrow E[aX] = aE[X]$$

$$a=0 \Rightarrow E[b] = b$$

<u>Defn</u>

The n^{th} <u>moment</u> of X

(for $n = 0, 1, 2, \ldots$) is

$$E[X^n]$$

Thus, the mean $E[X]$ is the 1^{st} moment.

<u>Defn</u>

The <u>variance</u> of X is the mean of the r.v. $(X - E[X])^2$

<u>note</u>: $(X - E[X])^2$ is non-negative, so necessarily $Var(X) \geq 0$

so

$$\mathrm{Var}(X) = E\left[(X - E[X])^2\right]$$

Defn

The standard deviation of X is

$$\sigma_X = \sqrt{\mathrm{Var}(X)}$$

Ex. (Previous)

$$P_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & x = 1 \\ .5 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

We computed

$$E[X] = 1.1 \qquad (1^{st} \text{ moment})$$

$$E[X^2] = 2.5 \qquad (2^{nd} \text{ moment})$$

now

$$Var(X) = E[(X-(1.1))^2]$$

$$= \sum_x (x-(1.1))^2 \cdot P_X(x)$$

$$= (-1-(1.1))^2 \cdot (.2) + (1-(1.1))^2 (.3) + (2-1.1)^2 (.5)$$

$$= \boxed{1.29}$$

and

$$\sigma_X = \boxed{1.1358}$$

Variance in terms of moments

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

Proof

notation: $\mu_X = E[X]$. Then

$$\mathrm{Var}(X) = E[(X-\mu_X)^2]$$

$$= \sum_x (x-\mu_X)^2 p_X(x)$$

$$= \sum_x (x^2 - 2\mu_X x + \mu_X^2) p_X(x)$$

$$= \sum_x x^2 p_X(x) - 2\mu_X \sum_x x\, p_X(x) + \mu_X^2 \sum_x p_X(x)$$

$$= E[X^2] - 2\mu_X \cdot \mu_X + \mu_X^2 \cdot 1$$

$$= E[X^2] - \mu_X^2$$

$$= E[X^2] - E[X]^2$$

Ex. (Previous)

We had $E[X] = 1.1$, $E[X^2] = 2.5$,

so

$$Var(X) = (2.5) - (1.1)^2$$

$$= \boxed{1.29}$$

Ex. $Y = aX + b$.

$$E[Y] = aE[X] + b$$

find 2nd moment of Y

$$E[Y^2] = E[(aX+b)^2]$$

$$= \sum_x (ax+b)^2 \cdot P_X(x)$$

$$\vdots \quad \boxed{\text{exercise}}$$

$$= a^2 E[X^2] + 2ab\,E[X] + b^2$$

now

$$Var(Y) = E[Y^2] - E[Y]^2$$

$$= \left(a^2 E[X^2] + 2abE[X] + b^2\right) - \left(aE[X]+b\right)^2$$

$$\boxed{\text{exercise}}$$

$$= a^2 \left(E[X^2] - E[X]^2\right)$$

$$= a^2 Var(X)$$

<u>Summary</u>

- $E[aX+b] = aE[X]+b$
- $Var(aX+b) = a^2 Var(X)$

mean & Variance of common r.v.s

Ex. Bernoulli (p)

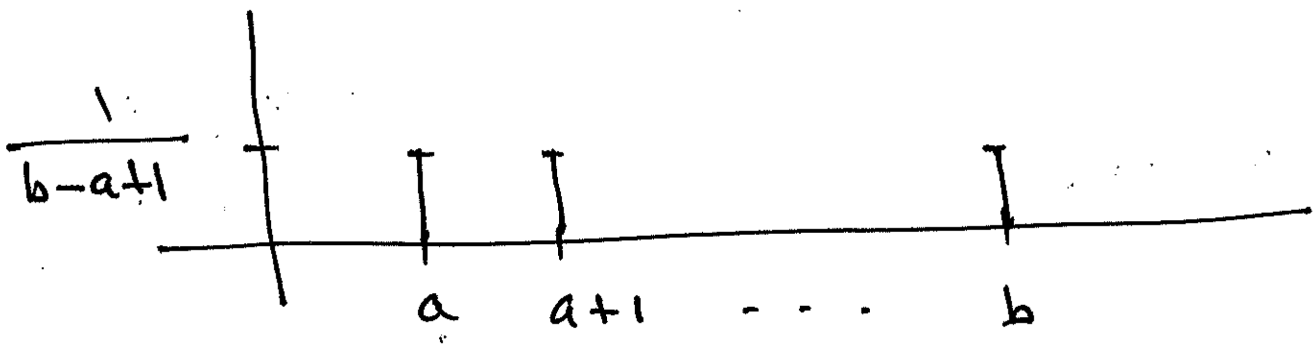
$$P_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \end{cases}$$

$$E[X] = 1 \cdot p + 0 \cdot (1-p) = p$$

$$E[X^2] = 1^2 \cdot p + 0^2 \cdot (1-p) = p$$

$$\begin{aligned} Var(X) &= E[X^2] - E[X]^2 \\ &= p - p^2 \\ &= p(1-p) \end{aligned}$$

Ex. Discrete Uniform r.v.



$$p_X(k) = \begin{cases} \dfrac{1}{b-a+1} & \text{if } k = a, a+1, \ldots, b \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{k=a}^{b} k \cdot \frac{1}{b-a+1}$$

$$= \left(\frac{1}{b-a+1}\right) \cdot \left(\sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k \right)$$

$$= \frac{1}{(b-a+1)} \cdot \left(\frac{b(b+1)}{2} - \frac{a(a-1)}{2} \right)$$

$$= \frac{1}{2}\left(\frac{1}{b-a+1}\right)\left(b^2+b-(a^2-a)\right)$$

$$= \frac{1}{2}\left(\frac{1}{b-a+1}\right)\left(\underbrace{(b^2-a^2)}_{(b-a)(b+a)}+(b+a)\right)$$

$$= \frac{1}{2}\left(\frac{1}{\cancel{b-a+1}}\right)(a+b)(\cancel{b-a+1})$$

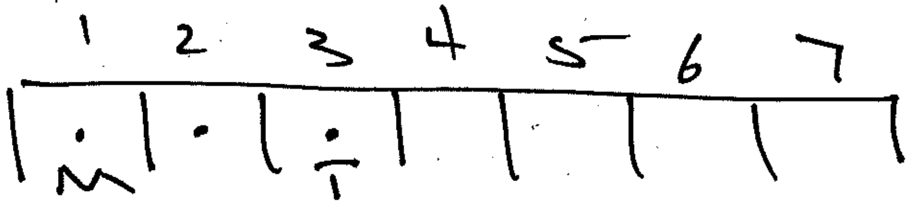
$$= \frac{a+b}{2}$$

need $E[X^2]$

Question on:

Problem #2, hw 2

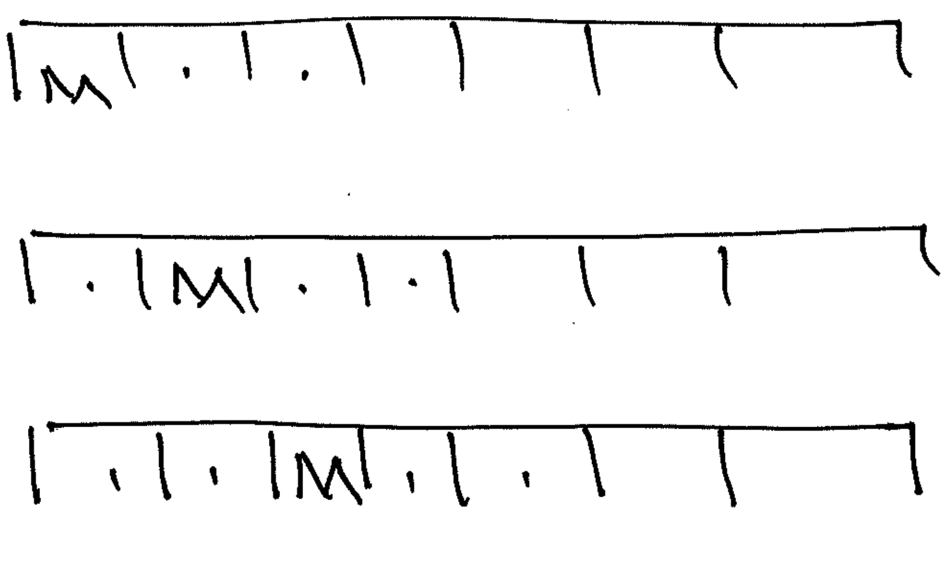
$n=7$



2 apart

~~means~~

1 empty between



3 terms in total Prob. thm.