CSE 107 4-11-25

Supplemental Lecture

The Geometric random variable

We perform Bernoulli trials, until the 1^{st} success.

$$X = \#\text{trials until } 1^{st} \text{ success}$$

note

$$\Omega = \{1, 01, 001, 0001, 00001, \ldots, 000\ldots0\}$$

so

$$p_X(k) = P(X=k) = (1-p)^{k-1} \cdot p$$

observe: $\text{range}(X) = \{1, 2, 3, \ldots\}$

i.e.

$$P_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1,2,3\ldots.. \\ 0 & \text{otherwise} \end{cases}$$

<u>note</u>:

$$\sum_{k=1}^{\infty} P_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} \cdot p$$

$$= p \cdot \sum_{k=0}^{\infty} (1-p)^k$$

$$= p \cdot \frac{1}{1-(1-p)} = p \cdot \frac{1}{p} = 1 \quad \checkmark$$

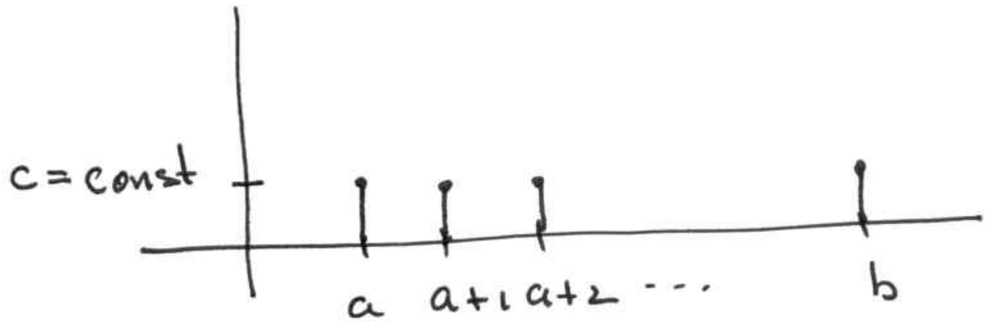
Parameter : p.

Discrete Uniform r.v. on $\{a, a+1, \dots, b\}$

$\text{range}(X) = \{a, a+1, a+2, \dots, b\}$

$a, b \in \mathbb{Z}$ with $a \le b$.

of #'s is $(b-a+1)$



$$p_X(k) = P(X=k) = c$$

must have $\displaystyle\sum_{k=a}^{b} p_X(k) = \sum_{k=a}^{b} c = 1$

so $\quad (b-a+1)\cdot c = 1$

$$\therefore \quad c = \frac{1}{b-a+1}$$

Thus

$$P_X(k) = \begin{cases} \dfrac{1}{b-a+1} & \text{if } k = a, a+1, \ldots, b \\ 0 & \text{otherwise} \end{cases}$$

The Poisson random variable

Parameter: λ

$$P_X(k) = \begin{cases} e^{-\lambda} \cdot \dfrac{\lambda^k}{k!} & \text{if } k = 0, 1, 2, \ldots\ldots \\ 0 & \text{otherwise} \end{cases}$$

observe

$$\sum_{k=0}^{\infty} P_X(k) = \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda} = e^{0} = 1 \quad \checkmark$$

The Poisson r.v. is like a Binomial r.v., counting # successes, but over infinitely many trials

$$P(\text{success}) = 0$$

more precisely

$$\boxed{e^{-\lambda} \cdot \frac{\lambda^k}{k!} \approx \binom{n}{k} p^k (1-p)^{n-k} \quad (k = 0, 1, 2 \dots)}$$

where $\lambda = np$, n is large, p is small.

even more precisely: let $\lambda > 0$ be constant, and set $p = \frac{\lambda}{n}$. Then

$$\boxed{\lim_{n \to \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} = e^{-\lambda} \cdot \frac{\lambda^k}{k!}}$$

Ex. let $n = 200$, $\lambda = 2$, $p = \frac{\lambda}{n} = .01$

The probability of $k = 4$ successes in 200 trials with $P(\text{success}) = .01$

$$\binom{200}{4}(.01)^4(.99)^{196} = .0902197$$

This is well approximated by

$$e^{-2} \cdot \frac{2^4}{4!} = .0902235$$

2.3 Functions of a random variable

Given a r.v. $X : \Omega \to \mathbb{R}$, we can compose X with a function

$$g : \mathbb{R} \to \mathbb{R}$$

i.e.

$$\Omega \xrightarrow{X} \mathbb{R} \xrightarrow{g} \mathbb{R}$$

The result is another random variable

$$Y = g \circ X = g(X) : \Omega \to \mathbb{R}$$

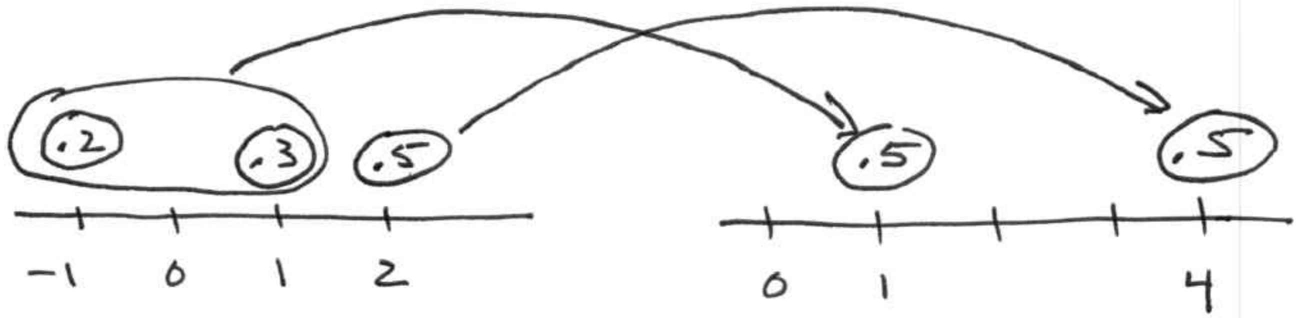
Ex. Suppose

$$P_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & \text{if } x = 1 \\ .5 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

let $Y = X^2$ (i.e. $g(x) = x^2$). Find the PMF of Y. note $\text{range}(Y) = \{1, 4\}$

$$
\begin{aligned}
P_Y(y) &= P(Y=y) \\
&= P(X^2=y) \\
&= P(X=\pm\sqrt{y}) \\
&= P(X=\sqrt{y} \text{ or } X=-\sqrt{y}) \\
&= P(\{X=\sqrt{y}\} \cup \{X=-\sqrt{y}\}) \\
&= P(X=\sqrt{y}) + P(X=-\sqrt{y}) \\
&= \begin{cases} \cancel{.2 \quad \sqrt{y}=-1} \\ .3 \quad \sqrt{y}=1 \\ .5 \quad \sqrt{y}=2 \end{cases} + \begin{cases} .2 \quad -\sqrt{y}=-1 \\ \cancel{.3 \quad -\sqrt{y}=1} \\ \cancel{.5 \quad -\sqrt{y}=2} \end{cases} \\
&= \begin{cases} .3 & y=1 \\ .5 & y=4 \\ 0 & \text{otherwise} \end{cases} + \begin{cases} .2 & y=1 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} .5 & \text{if } y=1 \\ .5 & \text{if } y=4 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Picture



$$\mathbb{R} \xrightarrow[g(x)=x^2]{g} \mathbb{R}$$