CSE 107 4-10-25

Binomial Theorem

If $n \geq 0$, $x, y \in \mathbb{R}$, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Binomial Coefficient

$$\binom{n}{k} = \left[\begin{matrix}\text{\# of } k\text{-subsets} \\ \text{of an } n\text{-set}\end{matrix}\right] = \left[\begin{matrix}\text{\# bit strings} \\ \text{of len. } n \\ \text{having } k \text{ 1's}\end{matrix}\right]$$

$$= \frac{n!}{k!\,(n-k)!}$$

If $n=3$:

$$(x+y)^3 = \underset{1}{(x+y)}\underset{2}{(x+y)}\underset{3}{(x+y)}$$

$$= xxx + xxy + xyx + \boxed{xyy} + yxx + \boxed{yxy} + \boxed{xyx} + yyy$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

# of subsets of $\{1,2,3\}$ of size 2:
$\{1,2\}, \{1,3\}, \{2,3\}$

bit strings of len. 3 containing exactly 2 1's

If we perform n independent Bernoulli trials with

$$P(\text{success}) = p$$

then

$$\boxed{P(\#\text{successes} = k) = \binom{n}{k} p^k (1-p)^{n-k}}$$

Binomial Probabilities

where $0 \le k \le n$.

1.6 Counting / Combinatorics

Summary

- # Permutations of n objects $= n!$

- # k-Permutations of n objects $(0 \le k \le n)$

$$= \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1)$$

- # k-combinations of n objects

$$= \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- # k-combinations of n objects, where repetition is allowed.

$$= \binom{n+k-1}{k}$$

Ex. why is $\binom{n}{k}$ = # bit strs. of len n with k 1's ?

let $n=4$, $k=2$

$S = \{1, 2, 3, 4\}$ Subsets

bit strs. →

0	0	0	0	→	$\emptyset$
0	0	0	1	→	$\{4\}$
0	0	1	0	→	$\{3\}$
0	0	1	1	→	$\{3,4\}$
0	1	0	0	→	$\{2\}$
0	1	0	1	→	$\{2,4\}$
0	1	1	0	→	$\{2,3\}$
0	1	1	1	→	$\{2,3,4\}$
1	0	0	0	→	$\{1\}$
1	0	0	1	→	$\{1,4\}$
1	0	1	0	→	$\{1,3\}$
1	0	1	1	→	$\{1,3,4\}$
1	1	0	0	→	$\{1,2\}$
1	1	0	1	→	$\{1,2,4\}$
1	1	1	0	→	$\{1,2,3\}$
1	1	1	1	→	$\{1,2,3,4\}$

<u>note</u>

$$\binom{4}{2} = \frac{4!}{2!\,(4-2)!} = \frac{\overset{2}{\cancel{4}} \cdot 3 \cdot \cancel{2 \cdot 1}}{\cancel{2} \cdot 1 \cdot \cancel{2 \cdot 1}} = \boxed{6}$$

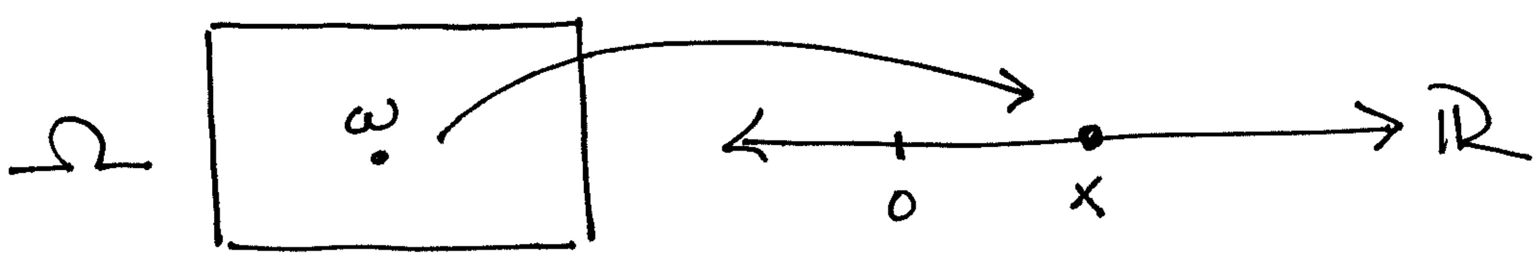
2.1 Random Variables

Defn

A <u>random variable</u> is a function

$$X : \Omega \to \mathbb{R}$$

(not random, not variable).

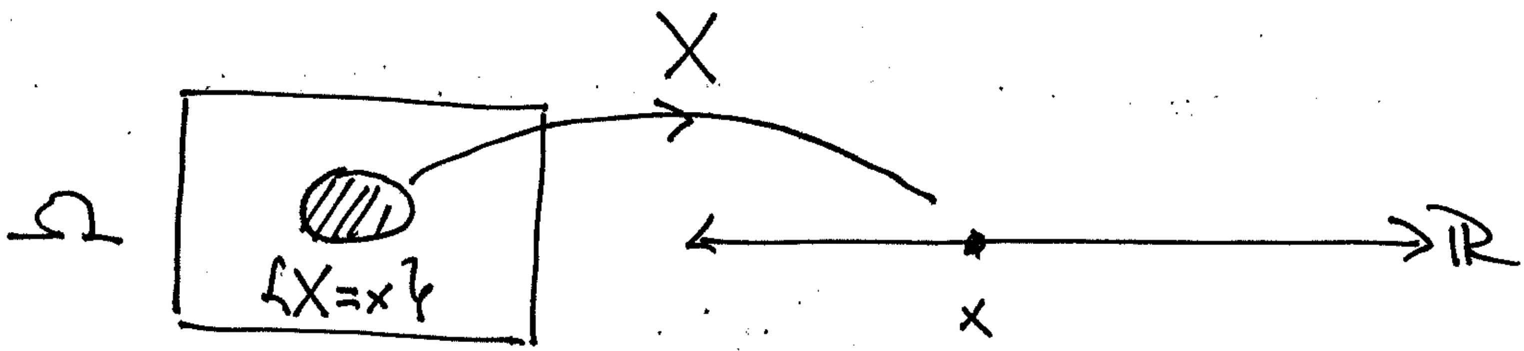


$$X(\omega) = x$$

capital   lower case

- notation:

$$\{X=x\}=\{\omega\in\Omega \mid X(\omega)=x\}\subseteq\Omega$$

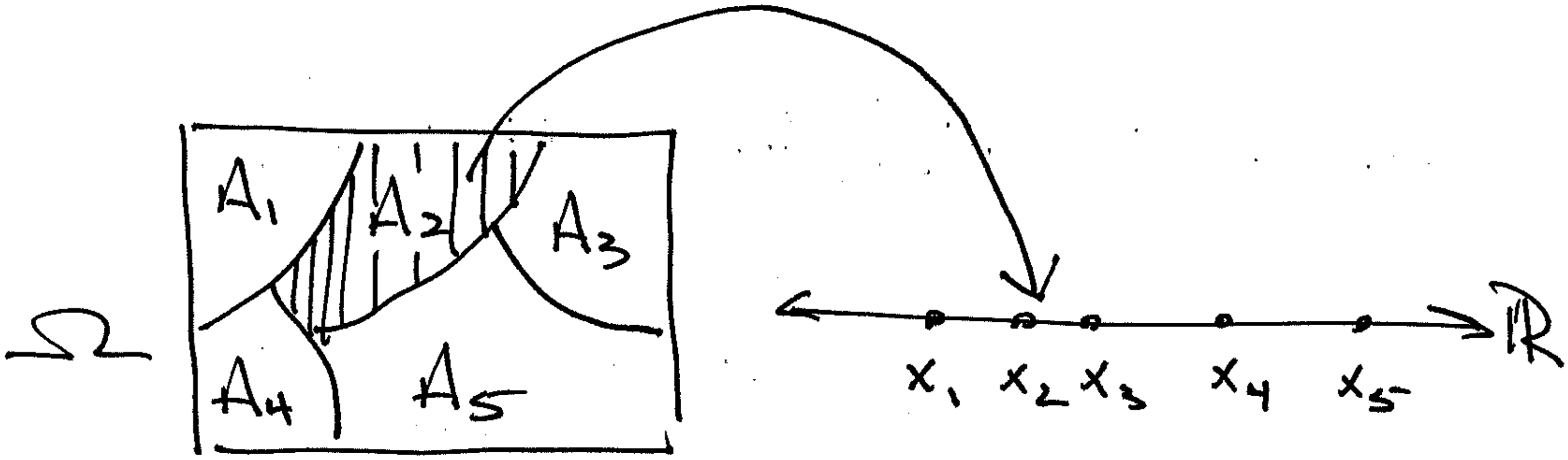


- Intuition: Probability is 'mass'.
- X is called <u>discrete</u> iff its range

$$\text{range}(X)=\{X(\omega) \mid \omega\in\Omega\}\subseteq\mathbb{R}$$

is a discrete subset of $\mathbb{R}$.

Picture



$$\Omega \xrightarrow[X]{} \mathbb{R}$$

$$A_2 = \{X = x_2\}$$

- X is called <u>continuous</u> ...... ?

Ex.

Roll 2 ~~fair~~ indep. dice. let

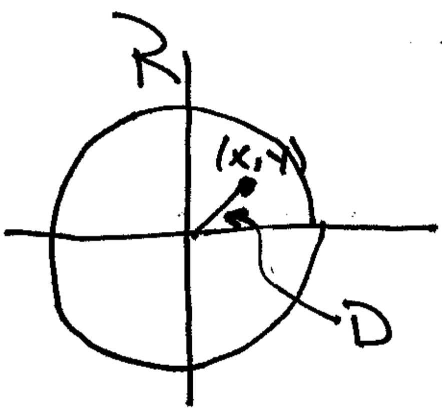
$$X((i,j)) = i+j$$

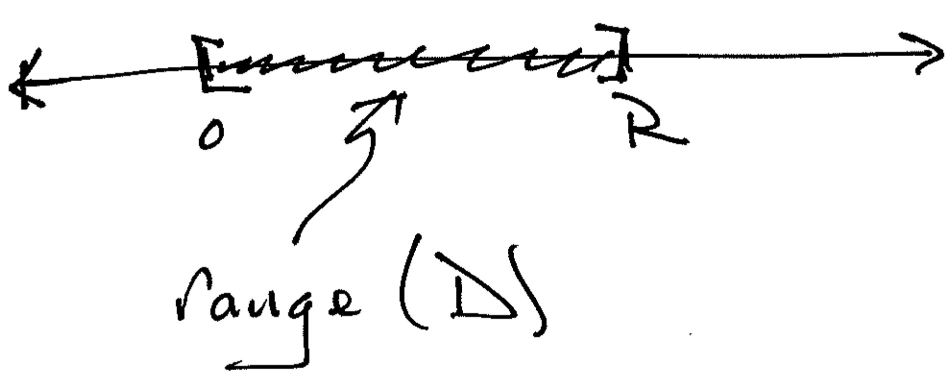
$$\underbrace{\begin{Bmatrix} 11 & 12 & \cdots & 16 \\ 21 & 22 & \cdots & 26 \\ \vdots & & & \vdots \\ 61 & 62 & \cdots & 66 \end{Bmatrix}}_{\Omega} \xrightarrow{X} \underbrace{\{2,3,4,5,6,7,8,9,10,11,12\}}_{\text{range}(X)}$$

Ex. Throw a dart at a circular target centered at $(0,0)$. Let

$$D\big((x,y)\big) = \sqrt{x^2+y^2}$$

the distance of impact point (x,y) to center $(0,0)$.





D is a continuous r.v.

2.2 Probability Mass functions

Defn

let X be a discrete r.v.
The Probability mass function (PMF) of X is

$$p_X(x) = \mathbb{P}(\{X=x\}) = \mathbb{P}(X=x)$$

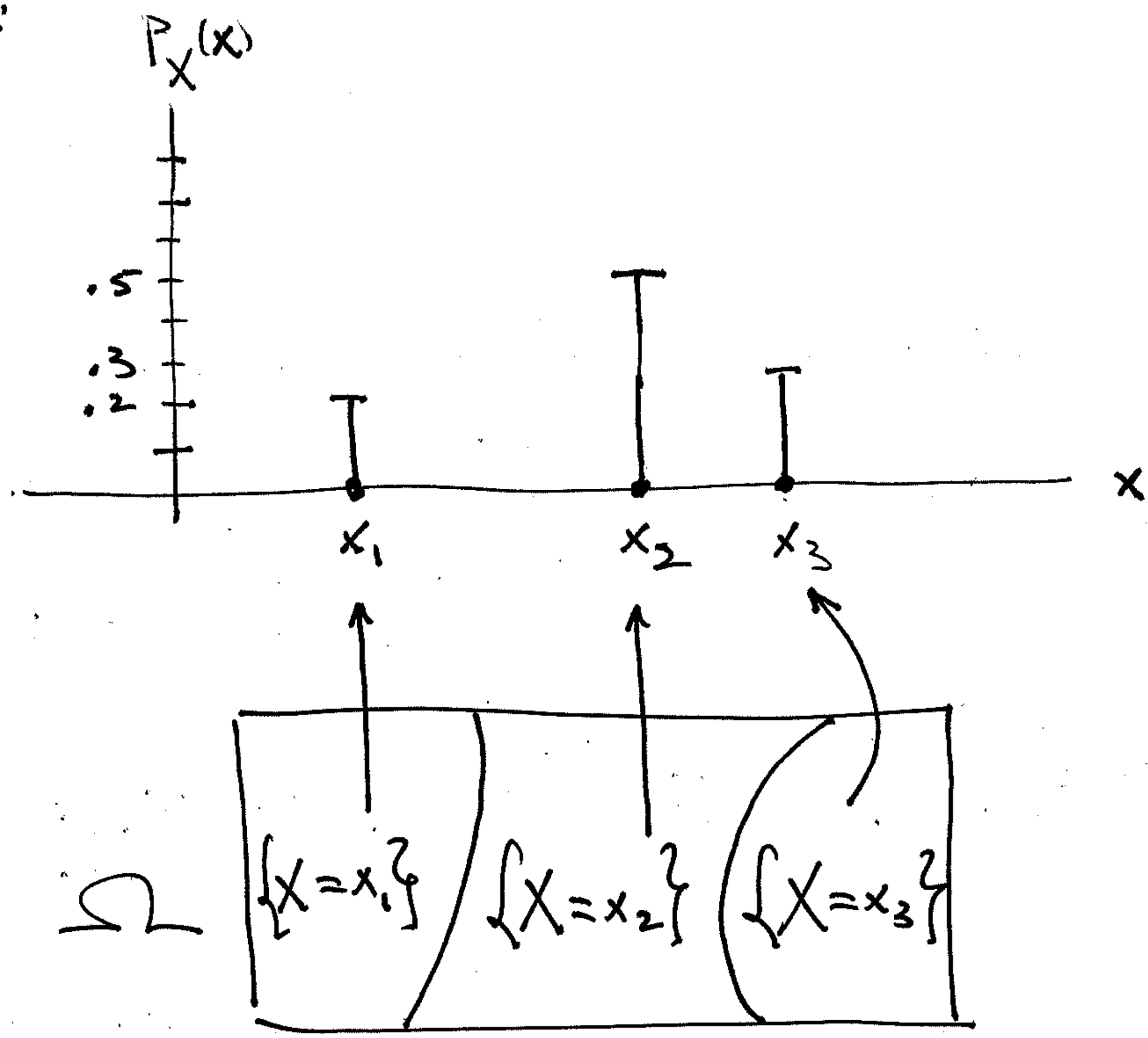
observe:

$$p_X : \mathbb{R} \longrightarrow [0,1]$$

also necessarily

$$\sum_{x \in \mathrm{range}(X)} p_X(x) = 1$$

Ex.



$$P_X(x) = \begin{cases} .2 & \text{if } x = x_1 \\ .5 & \text{if } x = x_2 \\ .3 & \text{if } x = x_3 \\ 0 & \text{otherwise} \end{cases}$$

The Bernoulli r.v.

Perform one Bernoulli trial with P(succ.) = p

and let

$$X(\omega) = \begin{cases} 1 & \text{if } \omega = \text{success} \\ 0 & \text{if } \omega = \text{failure} \end{cases}$$

Its PMF

$$P_X(k) = \begin{cases} p & \text{if } k = 1 \\ 1-p & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

We call p the <u>parameter</u>

The Binomial random variable

Perform a seq. of n independent Bernoulli trials, all with same parameter p.

$$\Omega = \{\text{bit strs. of len. } n\}$$

$$X(\omega) = \#\,\text{successes} = \#\,1\text{'s in } \omega$$

$$p_X(k) = P(X=k) = \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

i.e.

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

<u>note</u>:

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1^n = 1 \checkmark$$

The Binomial r.v. has 2 parameters: n = # trials
p = P(success).