CSE 107 4-1-25

1.1 Sets

Covered in CSE 16. Review this.

1.2 Probabilistic Models

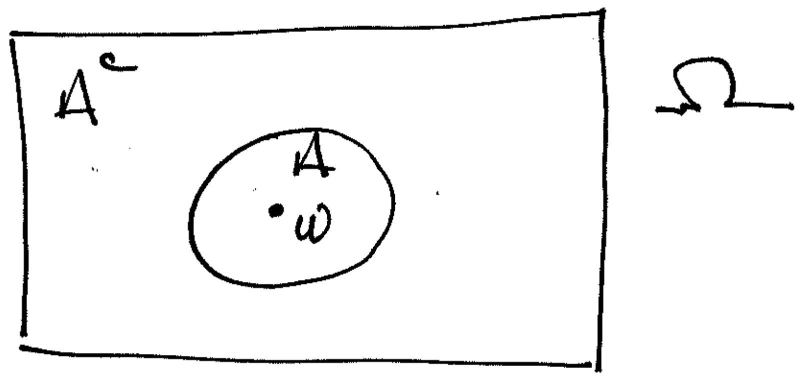
A Probability Model consists of 2 things.

(1) a set Ω called the sample space. The elements

$$\omega \in \Omega$$

are called outcomes of some experiment. A subset $A \subseteq \Omega$ is called an event.

(2) a <u>Probability law</u> P that assigns to each event $A \subseteq \Omega$ a real number $P(A)$ called the probability of A.



Axioms for $P(\cdot)$

(i) $P(A) \geq 0$ for all $A \subseteq \Omega$.

(ii) $P(\Omega) = 1$

(iii) additivity

- if $A, B \subseteq \Omega$ are disjoint (i.e. $A \cap B = \emptyset$), then

$$P(A \cup B) = P(A) + P(B)$$

<u>note</u> if $A, B, C \subseteq \Omega$ are pairwise disjoint (i.e. $A \cap B = A \cap C = B \cap C = \emptyset$)

$$P(A \cup (B \cup C)) = P(A) + P(B \cup C)$$
$$= P(A) + P(B) + P(C)$$

$\vdots$

any number of sets. called <u>finite additivity</u>.

- more generally, if $A_1, A_2, A_3, \ldots$ is an infinite sequence of pairwise disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots)$$

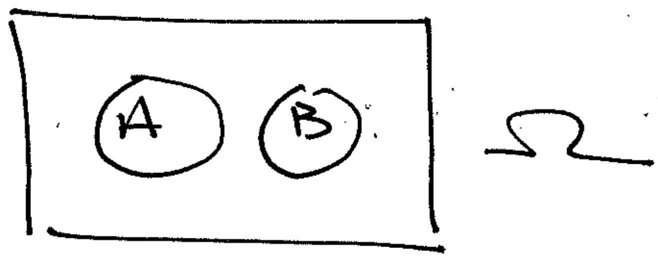
$$= P(A_1) + P(A_2) + P(A_3) + \ldots$$

i.e.

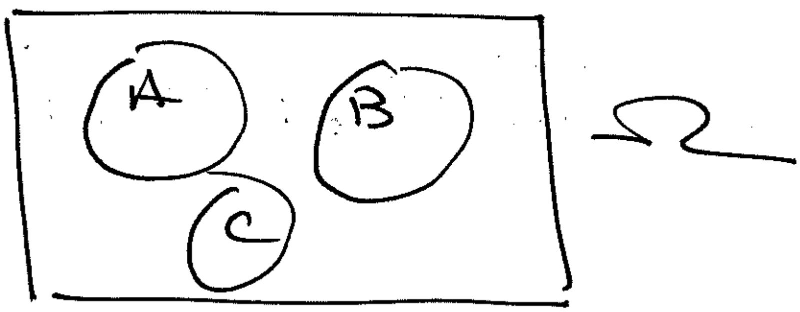
$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

called countable additivity.

2-sets

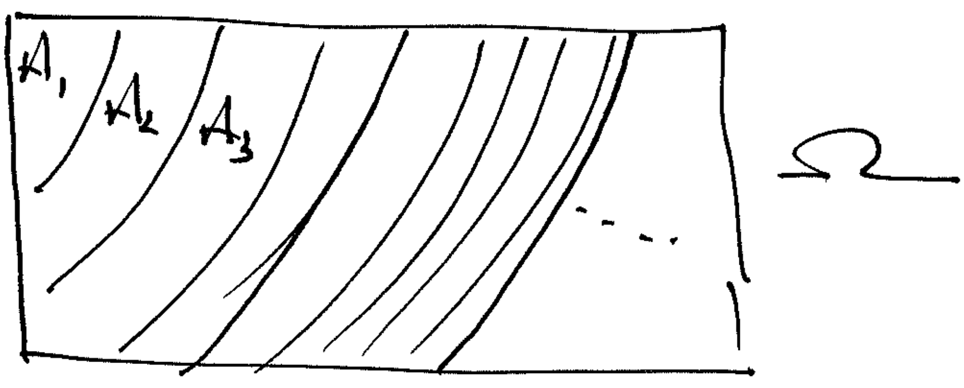


3-set



⋮

∞-many sets



Ex. flip a fair coin

$$\Omega = \{H, T\}$$

notation write $H = \{H\}$, $T = \{T\}$

events $\emptyset$, $\underbrace{\{H\}}_{H}$, $\underbrace{\{T\}}_{T}$, Ω

Since $\{H\} \cap \{T\} = \emptyset$, and $\{H\} \cup \{T\} = \Omega$

we have by axioms 2 and 3:

$$1 = P(\Omega) = P(H) + P(T)$$

fair means $P(H) = P(T)$. Thus

$$P(H) = P(T) = \frac{1}{2}$$

In general if $A \subseteq \Omega$, then

$$A \cup A^c = \Omega \quad \text{and} \quad A \cap A^c = \phi$$

Thus $P(A) + P(A^c) = P(\Omega) = 1$.

<u>note:</u> $P(A^c) = 1 - P(A)$

<u>or</u> $P(A) = 1 - P(A^c)$

<u>also</u> by axiom 2 $(P(A^c) \geq 0)$ we have $P(A) \leq 1$. Thus

$$0 \leq P(A) \leq 1$$

evidently P is a function

$$P : \underbrace{\mathbb{P}(\Omega)}_{\substack{\text{not really the}\\ \text{domain in all cases.}}} \to [0,1]$$

let $A = \Omega$ in the above. Then

$$1 = P(\Omega) = P(\Omega \cup \Omega^c)$$

$$= P(\Omega \cup \phi) = P(\Omega) + P(\phi)$$

$$= 1 + P(\phi)$$

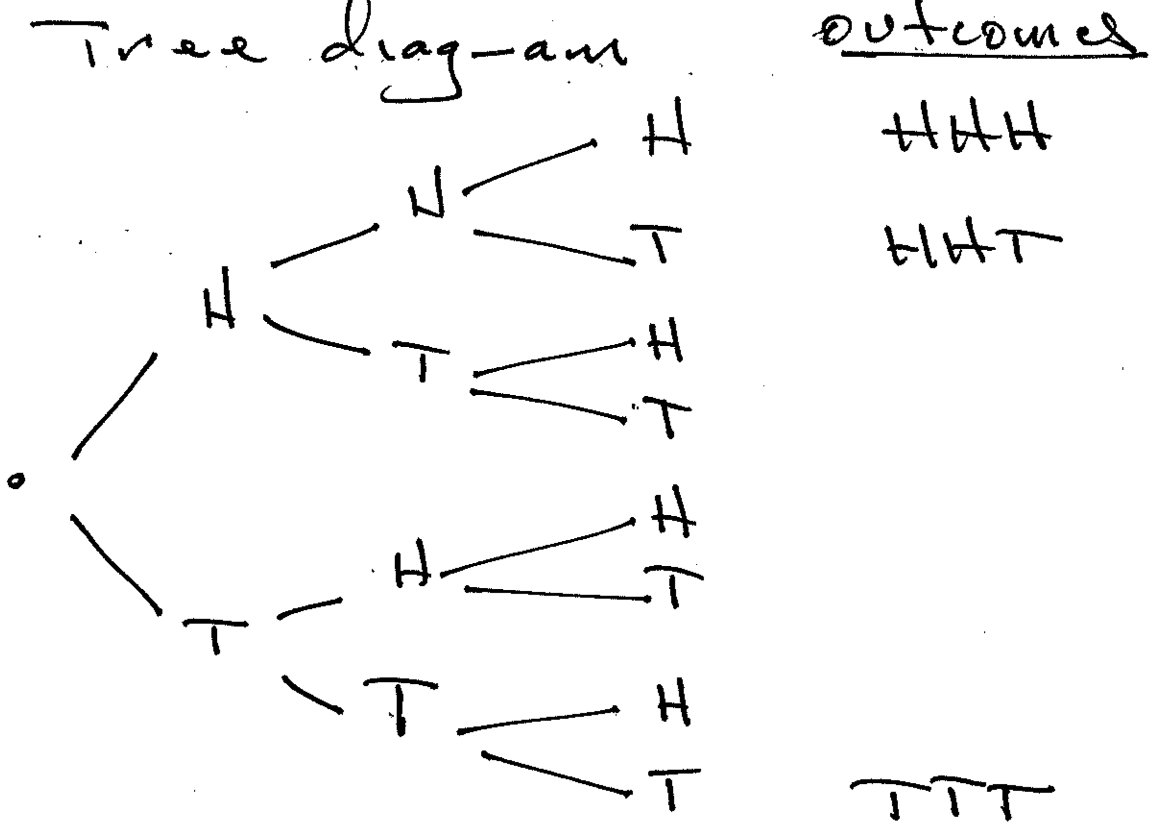
$\therefore P(\phi) = 0.$

Going back to $\Omega = \{H, T\}$, we have

$$\begin{array}{ccc} & P & \\ \Omega & \longrightarrow & 1 \\ H & \longrightarrow & \frac{1}{2} \\ T & \longrightarrow & \frac{1}{2} \\ \phi & \longrightarrow & 0 \end{array}$$

<u>EX</u> flip 3 fair coins. Then

$$\Omega = \{HHH, HHT, HTH, \underline{HTT}, THH, \underline{THT}, \underline{TTH}, TTT\}$$

Tree diagram 

Fair means all outcomes are equally likely, thus all have probability $\frac{1}{8}$. Let

$$A = \{\text{exactly 1 head}\}$$
$$= \{HTT, THT, TTH\}$$

since

$$A = \{HTT\} \cup \{THT\} \cup \{TTH\}$$

we have

$$P(A) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

<u>Discrete Probability law</u>

if Ω is finite, $P(\cdot)$ is specified by probabilities on outcomes. So if

$$A = \{\omega_1, \omega_2, \ldots, \omega_n\}$$

then

$$P(A) = P(\omega_1) + P(\omega_2) + \cdots + P(\omega_n)$$

Special case:

<u>Discrete uniform Probability law</u>

if Ω is finite, and if all $\omega \in \Omega$ are equally likely (i.e. have same Probability) then

$$P(A) = \frac{|A|}{|\Omega|}$$

Ex. two fair 6-sided dice are thrown. what is Ω ?

sum =	2	3	4	5	6	7	
	11	12	13	14	15	16	
	21	22	23	24	25	26	8
	31	32	33	34	35	36	9
	41	42	43	44	45	46	10
	51	52	53	54	55	56	11
	61	62	63	64	65	66	12

$P(\text{sum} = 2) = 1/36$

$P(\text{" " } 3) = 2/36$

$P(\text{" " } 4) =$