

CSE 107
Homework Assignment 3

Section 2.2: Probability mass functions

1. Alice and Bob play a chess match in which the first player to win a game wins the match. In each game $P(\text{Alice wins}) = 0.5$, $P(\text{Bob wins}) = 0.4$, and $P(\text{draw}) = 0.1$, and the outcomes of each game are independent. If 10 successive games are drawn, then the match is declared to be drawn.
 - (a) Determine the probability that Alice wins the match.
 - (b) Let X be the duration of the match, i.e. the number of games played. Determine the PMF of X .

Section 2.4: Expectation, mean and variance

2. A class with 300 students consists of 250 undergraduate students and 50 graduate students. The probability that an undergraduate student gets an A is $1/3$, and the probability that a graduate student gets an A is $1/2$. Let X be the number of students who get an A. Determine the expectation $E[X]$.

Hint: define random variables

$$U_i = \begin{cases} 1 & \text{if undergraduate student } i \text{ gets an A} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } 1 \leq i \leq 250$$

and

$$G_j = \begin{cases} 1 & \text{if graduate student } j \text{ gets an A} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } 1 \leq j \leq 50$$

Write X in terms of the U_i and G_j , then use linearity of expectation.

3. Let X be a random variable that takes integer values and is symmetric, that is,

$$P(X = k) = P(X = -k)$$

for all integers k . What is the expected value of $Y = \cos(X\pi)$ and $Y = \sin(X\pi)$?

4. A particular binary data transmission and reception device is prone to some error when receiving data. Suppose that each bit is read correctly with probability p . Find the smallest value of p such that when 10,000 bits are received, the expected number of errors is at most 10.
5. Let N be a nonnegative integer-valued random variable. Show that

$$E[N] = \sum_{i=1}^{\infty} P(N \geq i)$$

6. Two coins are tossed simultaneously until their results are different, i.e. one is Heads and the other Tails. The first coin has $P(\text{Head}) = p$ and the second $P(\text{Head}) = q$. All tosses are assumed to be independent. Let X be the number of tosses. (Note: for purposes of counting tosses, a simultaneous toss of both coins counts as 1 toss.) Find the PMF, expected value and variance of X .

Section 2.5: Joint PMFs of multiple random variables

7. The UCSC football team wins any one game with probability p , and loses it with probability $1 - p$. Its performance in each game is independent of its performance in other games. Let L_1 be the number of losses before its first win, and let L_2 be the number of losses after its first win and before its second win. Find the joint PMF of L_1 and L_2 .
8. A class of n students takes a test in which each student gets an A with probability p , a B with probability q , and a grade below B with probability $1 - p - q$, independently of any other student. Let X and Y be the numbers of students that getting A's and a B's, respectively. Determine the joint PMF $p_{X,Y}(x, y)$.

Section 2.6: Conditioning

9. Alvin shops for probability books for K hours, where K is a random variable that is equally likely to be 1, 2, 3, or 4. The number of books N that he buys is random and depends on how long he shops according to the conditional PMF

$$p_{N|K}(n|k) = \begin{cases} \frac{1}{k} & \text{if } n = 1, \dots, k \\ 0 & \text{otherwise} \end{cases}$$

- Find the joint PMF of K and N .
- Find the marginal PMF of N .
- Find the conditional PMF of K given that $N = 2$.
- Find the conditional mean and variance of K , given that he bought at least 2 but no more than 3 books.
- The cost of each book is a random variable with mean \$30. What is the expected value of his total expenditure? Hint: Condition on the events $\{N = 1\}$, $\{N = 2\}$, $\{N = 3\}$ and $\{N = 4\}$ and use the total expectation theorem.

Section 2.7: Independence

10. A fair 6-sided die is rolled until each of the outcomes $\{1, 2, 3, 4, 5, 6\}$ occurs at least once, where each roll is independent. Let X be the number of rolls performed. Determine the expected value of X . (Hint: call a roll a *success* if the number on top of the die occurs for the first time, and let X_i denote the number of rolls after the i^{th} success, up to and including the $(i + 1)^{\text{st}}$ success, for $1 \leq i \leq 5$. For instance, if the sequence (6, 6, 2, 6, 1, 1, 5, 1, 5, 5, 4, 4, 4, 3) is rolled, then $X_1 = 2$, $X_2 = 2$, $X_3 = 2$, $X_4 = 4$, $X_5 = 3$ and $X = 14$. Express X in terms of the X_i , then show that each X_i is geometric and find its parameter.)