CSE 107
Homework Assignment 2

Section 1.5: Independence

1. A peculiar six-sided die has uneven faces. In particular, the faces showing 1 or 6 are 1×1.5 inches, the faces showing 2 or 5 are 1×0.4 inches, and the faces showing 3 or 4 are 0.4×1.5 inches. Assume that the probability of a particular face coming up is proportional to its area. We independently roll the die twice. What is the probability that we get doubles?

2. A parking lot consists of a single row containing n parking spaces ($n \geq 2$). Mary arrives when all spaces are free. Tom is the next person to arrive. Each person makes an equally likely choice among all available spaces at the time of arrival. Describe the sample space. Obtain P(A), the probability the parking spaces selected by Mary and Tom are at most 2 spaces apart. (Note: this means that either Tom's space is next to Mary's, or that there is a single space between them.)

3. A particular jury consists of 7 jurors. Each juror has a 0.2 chance of making the wrong decision, independently of the others. If the jury reaches a decision by majority rule, what is the probability that it will reach a wrong decision?

4. Suppose that events A, B, and C are independent. Use the definition of independence to show that A and $B \cup C$ are independent.

5. A 7-sided die is thrown, with faces labeled $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$, and each outcome equally likely. Define events $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6\}$ and $C = \{1, 2, 4, 5, 6, 7\}$.

   (a) Determine whether A and B are independent.

   (b) Determine whether A and B are conditionally independent, given C.

6. A weighted coin has $P(H) = p$ and $P(T) = 1 - p$, where $0 < p < 1$. Suppose the coin is flipped indefinitely (i.e. infinitely many times), and assume all flips are independent. Define a_n to be the probability that after n flips, there have been an even number of heads.

   (a) Determine a_1, a_2 and a_3.

   (b) Determine a recurrence formula for a_n in terms of a_{n-1}, and p (i.e. find a relation of the form $a_n = f(a_{n-1}, p)$ where f is a function that you find explicitly.) Hint: Let A_n be the event that after n flips, there have been an even number of heads, then write an expression for $P(A_n)$ in terms of $P(A_{n-1})$ and $P(A_{n-1}^c)$.

   (c) Use the recurrence you found in part (b) to prove that

$$a_n = \frac{1}{2}(1 + (1 - 2p)^n)$$

Section 2.2: Probability mass functions

7. The annual premium of a special kind of insurance starts at \$1000 and is reduced by 10% after each year where no claim has been filed. The probability that a claim is filed in a given year is 0.05, independently of preceding years. What is the PMF of the total premium paid up to and including the year when the first claim is filed?

Section 2.3: Functions of a random variable

8. Let X be a discrete random variable that is uniformly distributed over the set of integers in the range $[a, b]$, where a and b are integers with $a < 0 < b$. Find the PMF of the random variables $\max(0, X)$ and $\min(0, X)$.

9. Let X be a discrete random variable, and let $Y = |X|$.

   (a) Assume that the PMF of X is

$$p_X(x) = \begin{cases} cx^2 & \text{if } x = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

   where c is a suitable constant. Determine the value of c.

   (b) For the PMF of X given in part (a) calculate the PMF of Y.

   (c) Give a general formula for the PMF of $Y = |X|$ in terms of the PMF of X.