CSE 107 9-30-25

continue ....

find

$P(\text{exactly } 1 \text{ head})$

$= P(\{HTT, THT, TTH\})$

$= P(\{HTT\} \cup \{THT\} \cup \{TTH\})$

$= P(HTT) + P(THT) + P(TTH)$

$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \boxed{\frac{3}{8}}$

Generalize

Discrete Probability Law

$P(\cdot)$ is seefied by Probabilities of singleton events $\omega \in \Omega$.

$$P(\omega) \text{ for all } \omega \in \Omega.$$

say $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$.

Then

$$P(\{\omega_5, \omega_{10}, \omega_{15}\})$$

$$= P(\omega_5) + P(\omega_{10}) + P(\omega_{15})$$

Special case:-

Discrete Uniform Probability law

all outcomes are equally likely,

i.e. each $\omega \in \Omega$ has

$$P(\omega) = \frac{1}{|\Omega|}$$

so if $A \subseteq \Omega$, then

$$P(A) = \frac{|A|}{|\Omega|}.$$

Ex. 2 6-sided fair dice are thrown. What is Ω?

SUM = 2 3 4 5 6 7

11	12	13	14	15	16	
21	22	23	24	25	26	8
31	32	33	34	35	36	9
41	42	43	44	45	46	10
51	52	53	54	55	56	11
61	62	63	64	65	66	12

max = 4

$$P(\text{sum} = 7) = \frac{6}{36} = \frac{1}{6}$$

$$P(\text{Sum is even}) = \frac{18}{36} = \frac{1}{2}$$

$$P(\max = 4) = \frac{7}{36}$$

$$P(\max \leq 4) = \frac{16}{36} = \frac{4}{9}$$

Continuous models

If Ω is some kind of continuum, then the singleton probabilities are not sufficient to determine $P(\cdot)$

EX R. and J. have a date, each will be delayed by a random time in range 0 to 1 hour. First to arrive will wait 15 minutes, then leave.

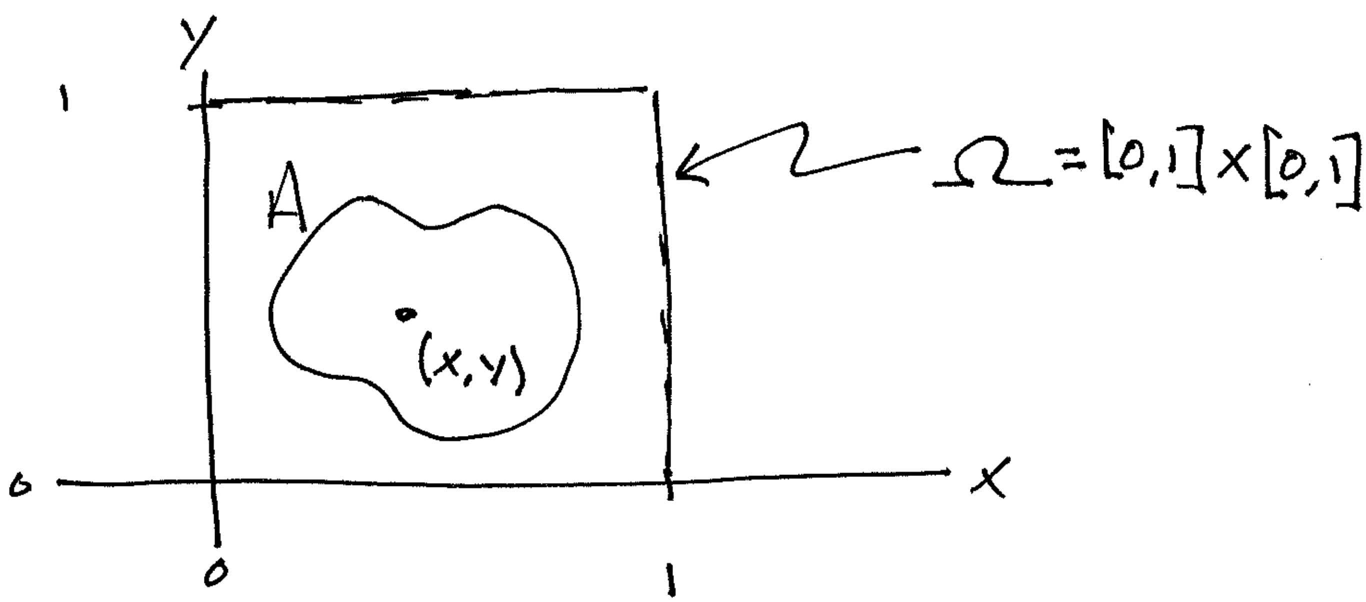
What is P(they meet) ?

Let

x = R's delay time

y = J's " "

so $0 \le x \le 1$ and $0 \le y \le 1$



equally likely means

$$P(A) = \text{const} \cdot \text{area}(A)$$

$$1 = P(\Omega) = \text{const} \cdot \text{area}(\Omega)$$
$$= \text{const} \cdot 1$$

$\therefore$ const $= 1$

$\therefore P(A) = \text{area}(A)$

<u>note</u>: $P((x,y)) = 0$, since area of a point is 0.

We seek $P(M)$ where

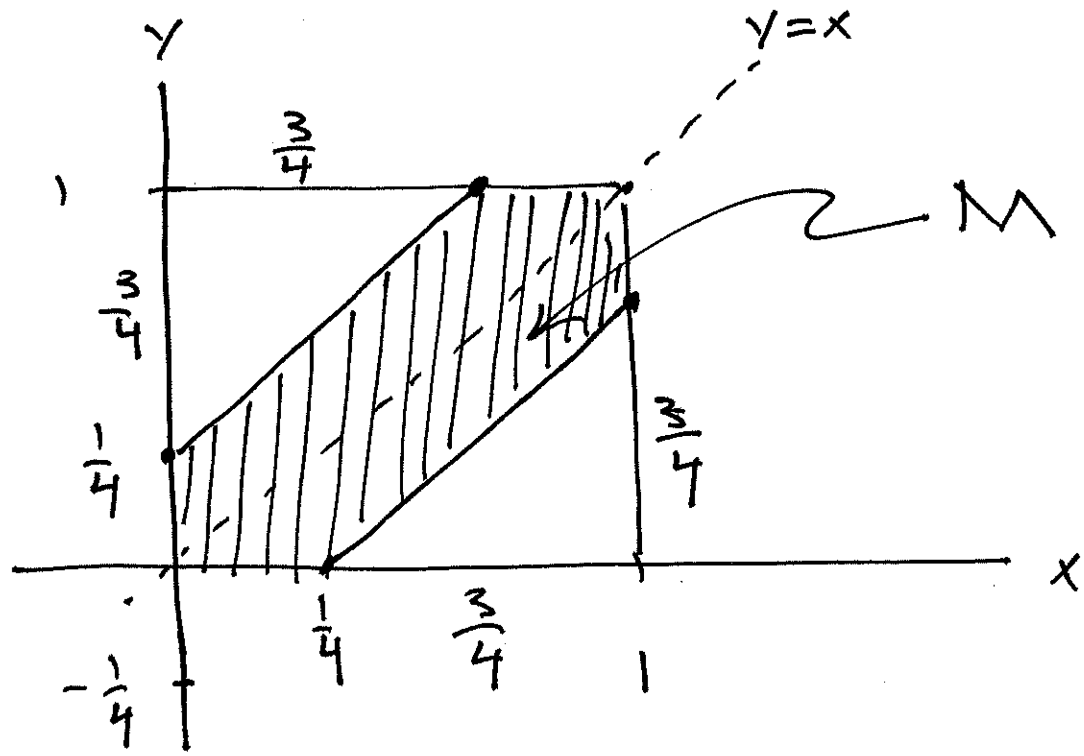
$$M = \{(x,y) \mid |x-y| \le \tfrac{1}{4},\ (x,y) \in \Omega\}$$

Observe $(x,y) \in M$ iff

$$|x-y| \le \frac{1}{4}$$

$$\therefore \quad -\frac{1}{4} \le x-y \le \frac{1}{4}$$

$$\therefore \quad y \le x+\frac{1}{4} \text{ and } y \ge x-\frac{1}{4}$$



$$\therefore \quad P(M) = 1-\left(\frac{3}{4}\right)^2 = 1-\frac{9}{16} = \boxed{\frac{7}{16}}$$

Exercise

Alice & Bob each pick a random number in $[0,2]$. Let Alice's be x, Bob's y. What is the probability that both

$$2x > y^2 \text{ and } 2y > x^2$$

Hint find area of

$$\{(x,y) \in [0,2] \times [0,2] \mid 2x > y^2 \text{ and } 2y > x^2\}$$

Properties of $P(\cdot)$

We can Prove

(a) if $A \subseteq B$, then $P(A) \leq P(B)$

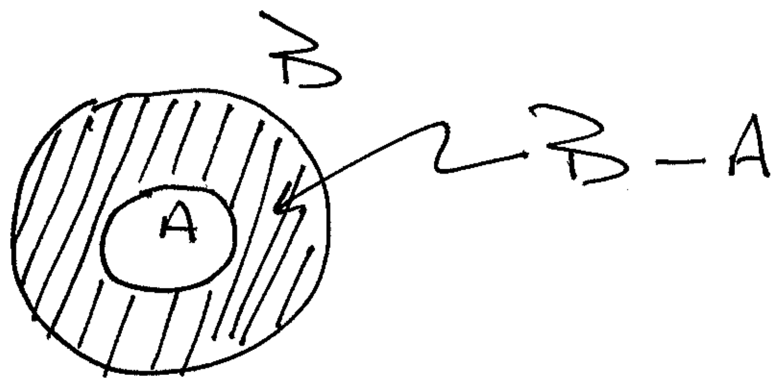
(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(c) $P(A \cup B) \leq P(A) + P(B)$

(d) $P(A \cup B \cup C) = P(A) + P(A^c \cap B)$
$+ P(A^c \cap B^c \cap C)$

Proof

(a) $A \subseteq B \Rightarrow \begin{cases} B = A \cup (B-A) \\ A \cap (B-A) = \phi \end{cases}$



By axiom (iii) we have

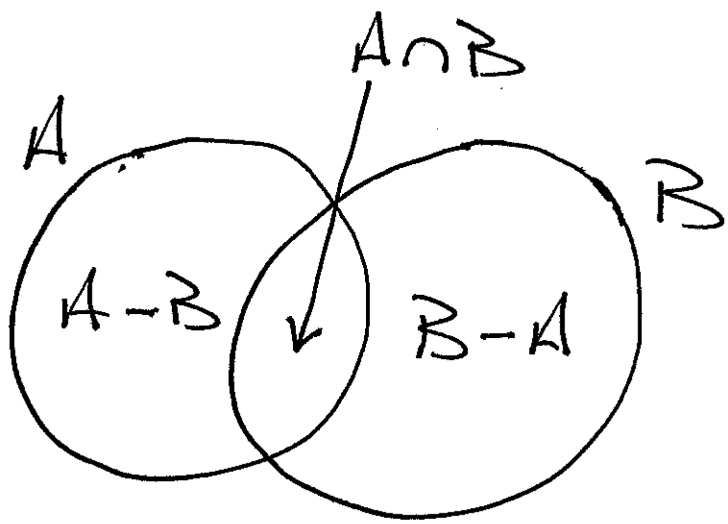
$$P(B) = P(A) + P(B-A) \geq P(A)$$

$$\therefore \; P(A) \leq P(B).$$

$\blacksquare$

(b) Proof: by axiom iii

- $P(A \cup B) = P(A-B) + P(A \cap B) + P(B-A)$



Also

$$P(B) = P(B-A) + P(A \cap B)$$

- $P(B-A) = P(B) - P(A \cap B)$

similarly

- $P(A-B) = P(A) - P(A \cap B)$

Thus

$$P(A\cup B) = \left(P(A) - \cancel{P(A\cap B)}\right) + \cancel{P(A\cap B)}$$
$$+ \left(P(B) - P(A\cap B)\right)$$
$$= P(A) + P(B) - P(A\cap B)$$

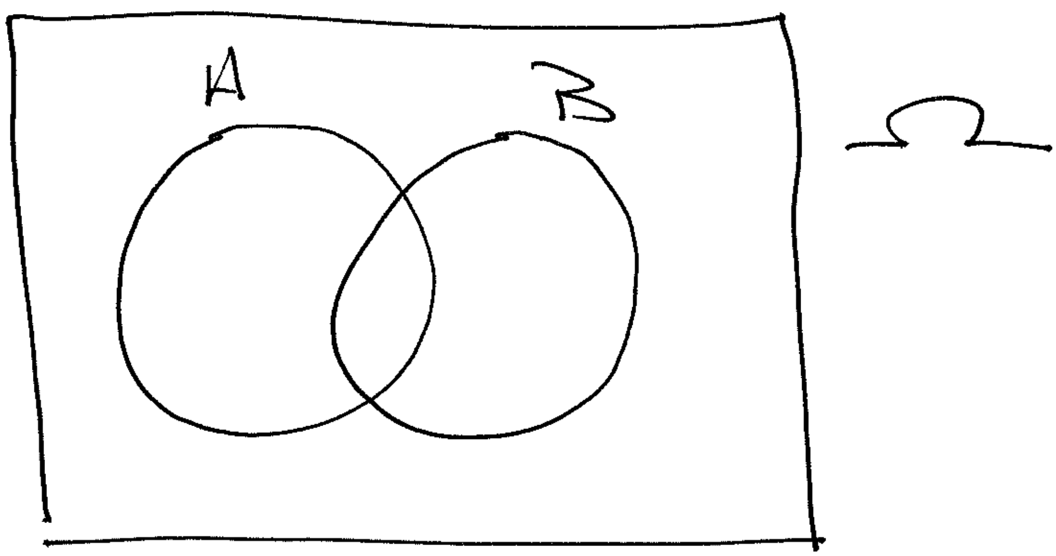
$\blacksquare$

(c) follows from (b).

(d) Prove this as an exercise.

1.3 Conditional Probability

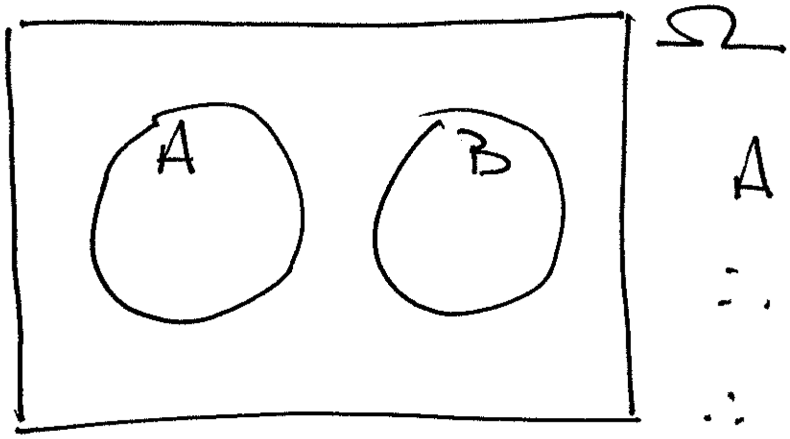
Suppose $A, B \subseteq \Omega$



If we are given the information that B occurred, it affects our knowledge/belief in Prob. that A occurred.

Two extreme cases:

1.)



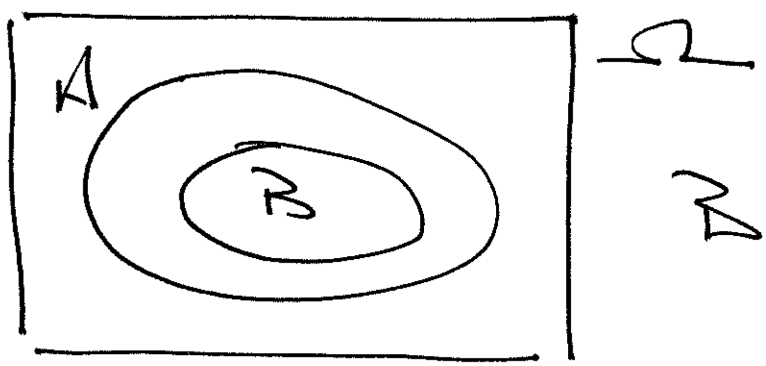
$A \cap B = \phi$

$\therefore A \subseteq B^c$

$\therefore B \subseteq A^c$

B occurrs $\Rightarrow$ A <u>does not</u> occur

2.)



$B \subseteq A$

B occurs $\Rightarrow$ A <u>does</u> occur

Defn

The <u>conditional Probability</u> of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

[We assume $P(B) > 0$]