CSE 107 12-2-25

Ex.

you receive email according to a Poisson Process at rate of

$$\lambda = 2 \, \text{msgs}/\text{hour}$$

you have no messages at $t=0$.

a) find Probability of 3 new messages at time $t = \frac{1}{2}$ hour.

$$P_{N_{\frac{1}{2}}}(3) = P(N_{\frac{1}{2}} = 3)$$

$$= e^{-2\cdot\frac{1}{2}} \cdot \frac{(2\cdot\frac{1}{2})^3}{3!}$$

$$= e^{-1} \cdot \frac{(1)^3}{3!} = e^{-1} \cdot \frac{1}{6} = \boxed{.0613}$$

b.) find Prob. of having no messages in 30 min.

$$P(N_{\frac{1}{2}} = 0) = P_{N_{\frac{1}{2}}}(0)$$

$$= e^{-(2 \cdot \frac{1}{2})} \cdot \frac{(2 \cdot \frac{1}{2})^0}{0!} = \boxed{e^{-1}}$$

$$= \boxed{.3679}$$

c.) what is expected time to 5^{th} new message

$$E[Y_5] = E[T_1 + T_2 + \cdots + T_5]$$

$$= 5 \cdot E[T_1]$$

$$= 5 \cdot \frac{1}{\lambda} = \boxed{\frac{5}{2}}$$

The sum of Poisson r.v.s

Let X, Y be indep. Poisson with param. λ_1, λ_2 resp.
we show $Z = X + Y$ is Poisson$(\lambda_1 + \lambda_2)$.

<u>Proof</u>

we know

$$P_X(k) = e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \quad (k = 0, 1, \ldots)$$

$$P_Y(k) = e^{-\lambda_2} \cdot \frac{\lambda_2^k}{k!} \quad (k = 0, 1, \ldots)$$

Thus

$$p_Z(n) = \sum_{k=0}^{\infty} p_X(k)\, p_Y(n-k)$$

$$= \sum_{k=0}^{n} e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \lambda_1^k \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot \lambda_1^k \lambda_2^{n-k}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^n}{n!} \quad (\text{any } n = 0, 1, \ldots)$$

which is Poisson$(\lambda_1+\lambda_2)$ ∎

Ex (6.2 #8)

During rush hour (8-9 am) accidents as a Poisson Process with rate 5/hour. During Period 9-11 am accidents occur as Poisson Process with rate 3/hour.

find PMF of total # of Accidents from 8-11 am?

Solution

let

N_1 = # accidents in 8–9 am

N_2 = " " " 9–11 am

and

$N = N_1 + N_2$ = # accidents in 8–11 am

N_1 is Poisson(5) and N_2 is Poisson(3·2) = Poisson(6)

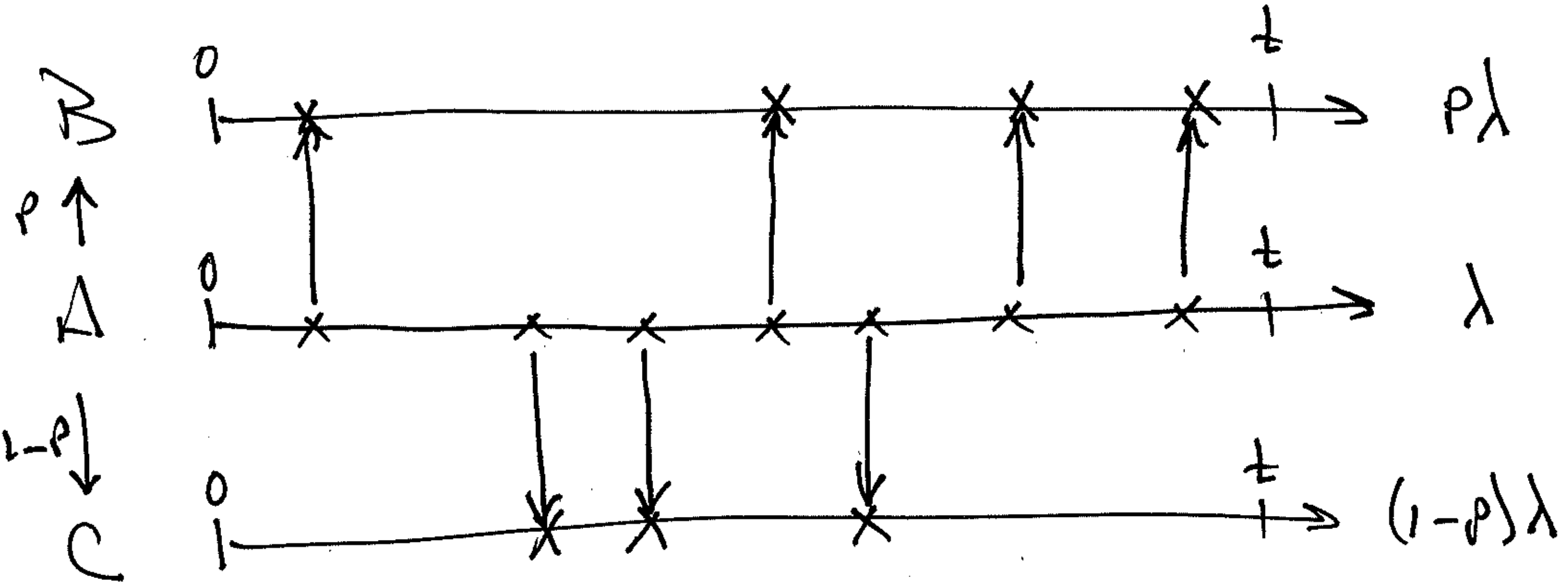
Thus N is Poisson(5+6)

i.e. Poisson(11). So

$$P_N(n) = e^{-11} \cdot \frac{11^n}{n!}$$

Splitting a Poisson Process

Let A be a Poisson Process of rate λ. Split A into 2 arrival Processes B, C with Prob. p, $1-p$ resp.



What kind of Processes are B and C?

let $N = \#$ arrivals in A in $[0, t]$

$X = \#$ " " B " "

$Y = \#$ " " C " "

we know N is Poisson(λt), so

$$P_N(n) = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} \quad (n = 0, 1, \ldots)$$

Then

$$P_X(k) = P(X = k)$$

$$= \sum_{n=k}^{\infty} P(X = k \mid N = n) \cdot P(N = n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{\cancel{n!}}{k!(n-k)!} p^k (1-p)^{n-k} e^{-\lambda t} \frac{(\lambda t)^n}{\cancel{n!}}$$

$$= \frac{p^k}{k!} \cdot e^{-\lambda t} \cdot \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} (\lambda t)^n}{(n-k)!}$$

substitute $\left. \begin{matrix} \\ n \leftarrow n+k \end{matrix} \right\}$

$$= \frac{p^k}{k!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{(1-p)^n (\lambda t)^{n+k}}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{((1-p) \cdot \lambda t)^n}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\lambda t} \cdot e^{(1-p)\lambda t}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\cancel{\lambda t} + \cancel{\lambda t} - p\lambda t}$$

$$= e^{-p\lambda t} \cdot \frac{(p\lambda t)^k}{k!} \quad (k = 0, 1, \ldots)$$

$\therefore$ X is Poisson$(p\lambda t)$

$\therefore$ B is a Poisson Process with rate $p\lambda$.

see examples on P. 318

Merging Poisson Processes

Ex.

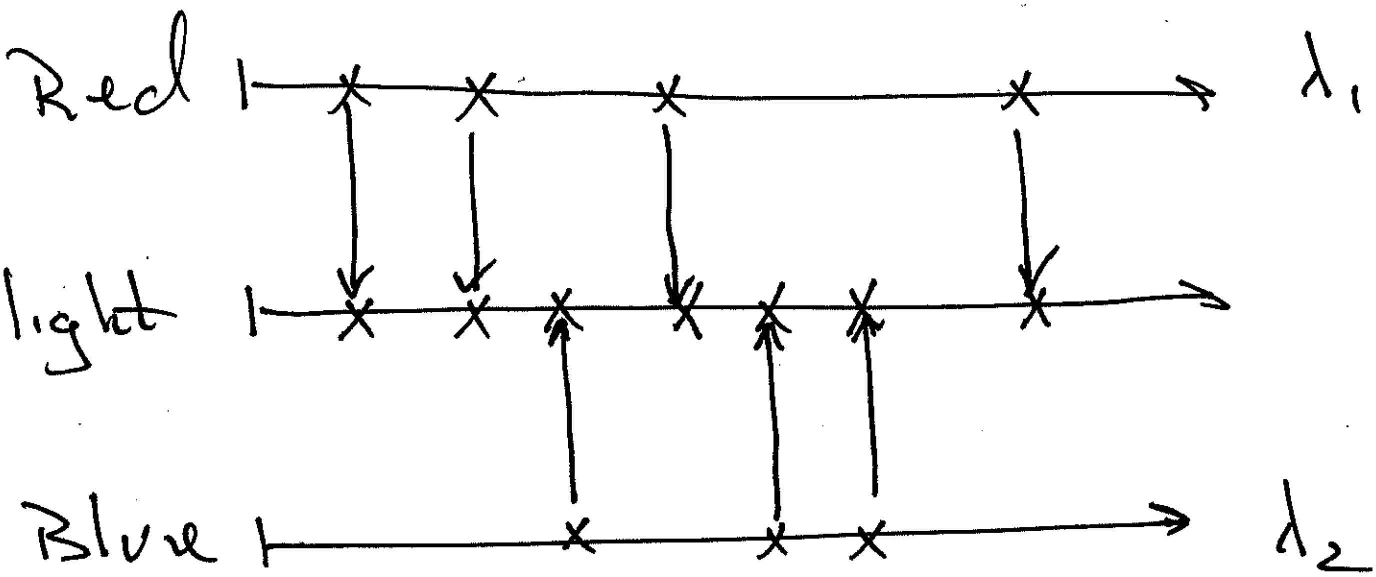
An observer sees 2 random flashing lights, red and blue.

Red is Poisson Processes rate λ_1

Blue " " " " λ_2.

The observer is colorblind and just sees lights.

What kind of Process does the observer see?



let

$X = \#$ Red lights in $[0, t]$

$Y = $ " Blue " " "

$M = $ # lights " "

$\therefore \quad M = X + Y$

$\therefore$ ---- Continue ------