By a similar argument C is also a Poisson Process of rate $(1-p)\lambda$.

Ex. see 6.7 #11

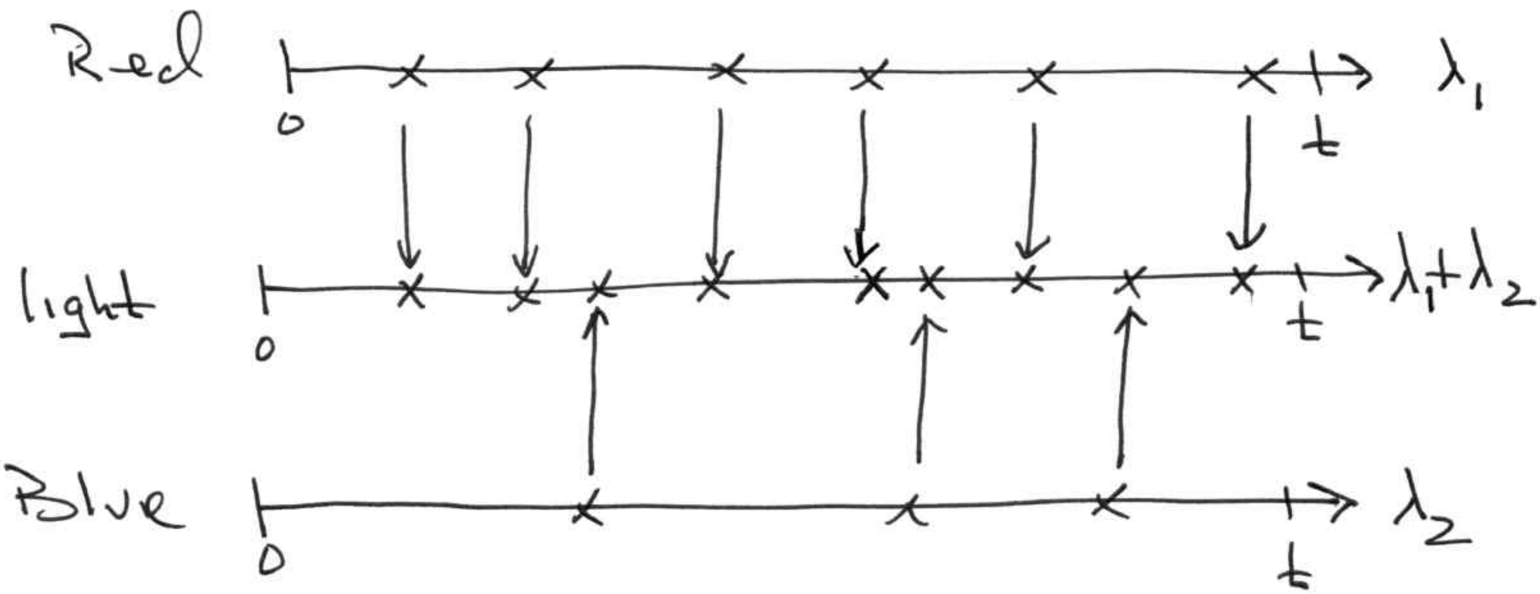
Merging Poisson Processes

Ex

an observer sees 2 lights flashing at random times, one red, one blue. The red light is a Poisson Process at rate λ_1, and blue is Poisson Processes of rate λ_2.

The observer is colorblind however, so he only sees lights.

what kind of process is the observer seeing?



let

X = # red lights in $[0,t]$

Y = " Blue " " "

M = # lights in $[0,t]$

Know X is Poisson$(\lambda_1 t)$

Y is Poisson$(\lambda_2 t)$

Thus

$$M = X + Y$$

is Poisson with Param $\lambda_1 t + \lambda_2 t = (\lambda_1 + \lambda_2) t$

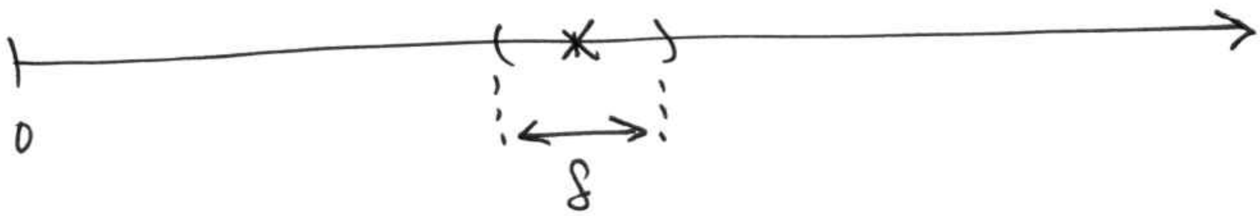
$\therefore$ M is a Poisson Process of rate $\lambda_1 + \lambda_2$.

<u>Ex</u>

Suppose observer sees a single flash. What is the Prob. that it was red?

<u>Solution 1</u>

let $\delta > 0$ be the width of a small time interval containing the flash



We seek

$$P(1\ \text{Red} \mid 1\ \text{light})$$

$$= \frac{P(1\ \text{Red},\ 1\ \text{light})}{P(1\ \text{light})}$$

$$= \frac{P(1\ \text{Red})}{P(1\ \text{light})} = \frac{\cancel{\delta} \cdot \lambda_1}{\cancel{\delta}(\lambda_1 + \lambda_2)} = \boxed{\frac{\lambda_1}{\lambda_1 + \lambda_2}}$$

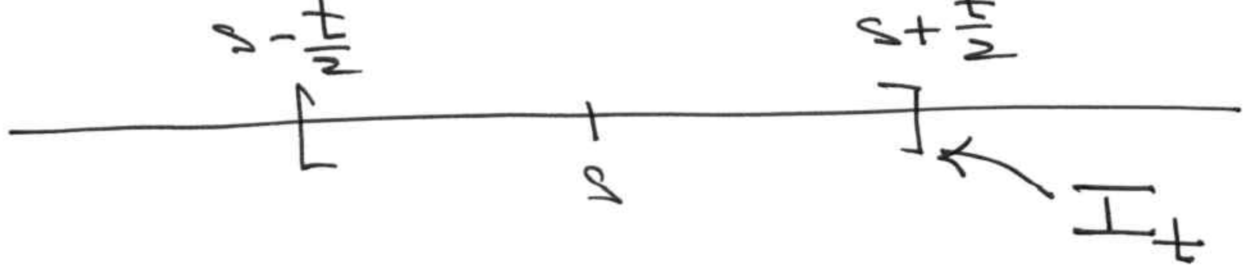
Similarly, the prob. that the colorblind observer saw a blue light is

$$P(1\ \text{blue} \mid 1\ \text{light}) = \boxed{\frac{\lambda_2}{\lambda_1 + \lambda_2}}$$

Solution 2:

(without small interval. Probabilities)

Suppose light flashes at time s. Consider an interval of length t centered at s.



Then

$$P(1 \text{ Red in } I_t \mid 1 \text{ light in } I_t)$$

$$= \frac{P(1 \text{ Red in } I_t,\ 1 \text{ light in } I_t)}{P(1 \text{ light in } I_t)}$$

$$\doteq \frac{P(1 \text{ Red in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{e^{-\lambda_1 t} \cdot \frac{(\lambda_1 t)^1}{1!}}{e^{-(\lambda_1+\lambda_2)t} \cdot \frac{((\lambda_1+\lambda_2)t)^1}{1!}}$$

$$= e^{\lambda_2 t} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} \xrightarrow[t \to 0]{\text{as}} \boxed{\frac{\lambda_1}{\lambda_1+\lambda_2}}$$

$\blacksquare$

7.1 Discrete Markov Chains

let S be a set of states

$$S = \{1, 2, 3, \ldots, m\}$$

let $X_0, X_1, X_2, \ldots$ be a seq. of r.v.s taking values in S.

consider

$$X_n = \text{state of process at time } n.$$

Define (<u>transition Probabilities</u>)

$$p_{ij} = P(X_{n+1} = j \mid X_n = i)$$

Markov Property

$$P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0)$$

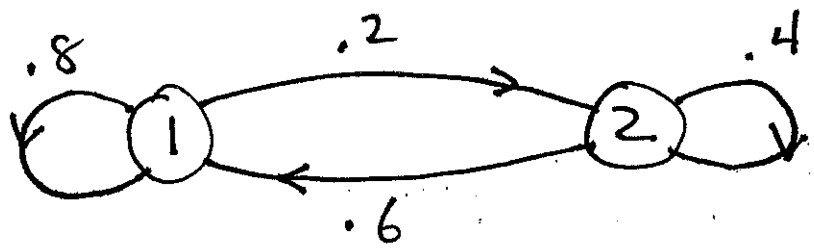
$$= P(X_{n+1}=j \mid X_n=i)$$

Defn

A Markov chain model consists of S, P_{ij} and the Markov Property.

Ex. $m = 2$, $S = \{1, 2\}$

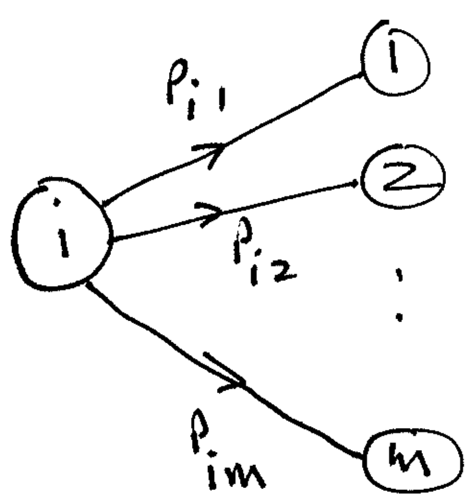
$p_{11} = .8$, $p_{12} = .2$, $p_{21} = .6$, $p_{22} = .4$



state transition diagram

note. transition Probabilities satisfy

$$\sum_{j=1}^{m} p_{ij} = 1 \quad (\text{for all } i \in S)$$



Suppose $\underline{\underline{X_0 = 1}}$. Find $P(X_2 = 2)$

By total Prob. Thm

$$P(X_2=2) = P(X_2=2 \mid X_1=1) \cdot P(X_1=1)$$
$$+ P(X_2=2 \mid X_1=2) \cdot P(X_1=2)$$
$$= p_{12} \cdot P(X_1=1) + p_{22} \cdot P(X_1=2)$$

$$P(X_1=1) = P(X_1=1 \mid X_0=1) \cdot \overbrace{P(X_0=1)}^{1}$$
$$+ P(X_1=1 \mid X_0=2) \cdot \overbrace{P(X_0=2)}^{0}$$

$$= p_{11}$$

16

$$P(X_1=2) = P(X_1=2 \mid X_0=1)\cdot \overbrace{P(X_0=1)}^{1} + 0$$

$$= P_{12}$$

$$\therefore\ P(X_2=2) = P_{12}\cdot P_{11} + P_{22}\cdot P_{12}$$

$$= P_{11}\cdot P_{12} + P_{12}\cdot P_{22}$$

$$= (.8)(.2) + (.2)(.4) = .16 + .08 = \boxed{.24}$$

similarly

$$P(X_2=1) = P_{11}\cdot P_{11} + P_{12}P_{21}$$

$$= P_{11}^2 + P_{12}P_{21} = \ldots = \boxed{.76}$$

In the same manner, if $X_0 = 2$,

then

$$P(X_2 = 1) = P_{21}P_{11} + P_{22}P_{21} = \cdots = \boxed{.72}$$

and

$$P(X_2 = 2) = P_{21}P_{12} + P_{22}P_{22} = \cdots = \boxed{.28}$$

To summarize, define <u>transition</u> <u>probability matrix</u>

$$R = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix}$$

L8

observe

$$P^2 = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

$$= \begin{pmatrix} p_{11}^2 + p_{12}p_{21} & p_{11}p_{12} + p_{12}p_{22} \\ p_{21}p_{11} + p_{22}p_{21} & p_{21}p_{12} + p_{22}^2 \end{pmatrix} = \begin{pmatrix} .76 & .24 \\ .72 & .28 \end{pmatrix}$$

<u>Defn</u>

A <u>Probability matrix</u> is a matrix with pos. real entries in which rows add to 1.

<u>Exercise</u> show that the product of probability matricies is a probability matrix.

L9

Thus the ij^{th} element in R^2 is the Prob. of going from (i) to (j) in 2 steps

Defn

The n-step transition probabilities are

$$r_{ij}(n) = P(X_n = j \mid X_0 = i)$$

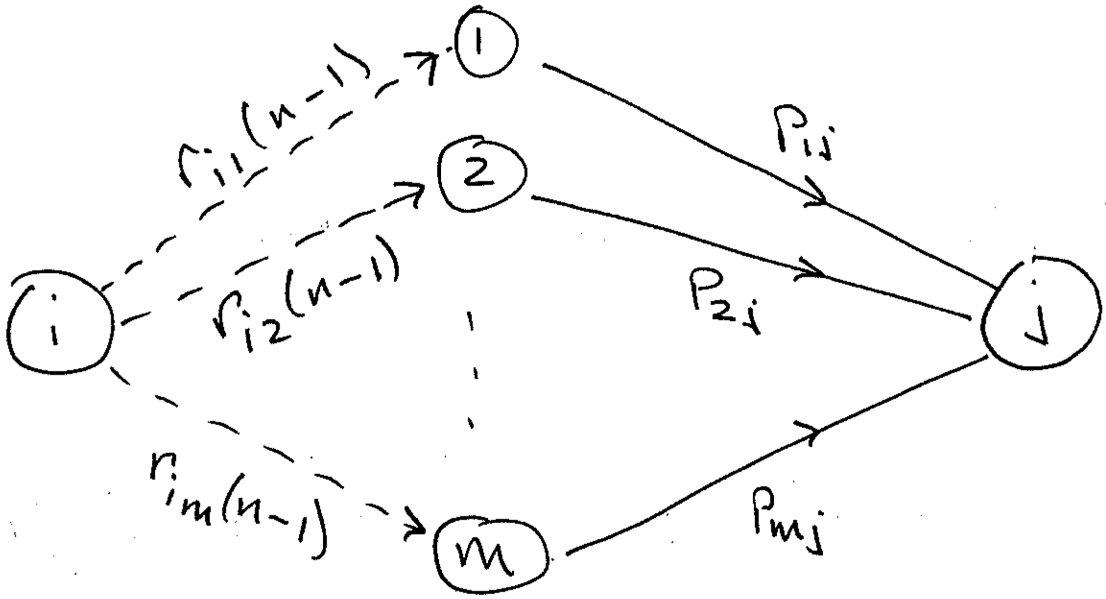
$$= P(X_{n+n_0} = j \mid X_{n_0} = i)$$

for any $i, j \in S$ and $n \geq 0$.

note $r_{ij}(1) = p_{ij}$.

Chapman-Kolmogorov Equations

$$\boxed{r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)\cdot p_{kj}}$$



Proof. apply total prob. thm, or see text.

matrix form of C.K. eqns.

$$\left(r_{ij}(n)\right)_{i,j\in S} = R^{n-1}\cdot R = R^{n}$$

Ex. Previous

$$R^{3} = \cdots = \begin{pmatrix} .752 & .248 \\ .744 & .256 \end{pmatrix}$$

so $r_{21}(3) = P(X_3 = 1 \mid X_0 = 2) = .744$

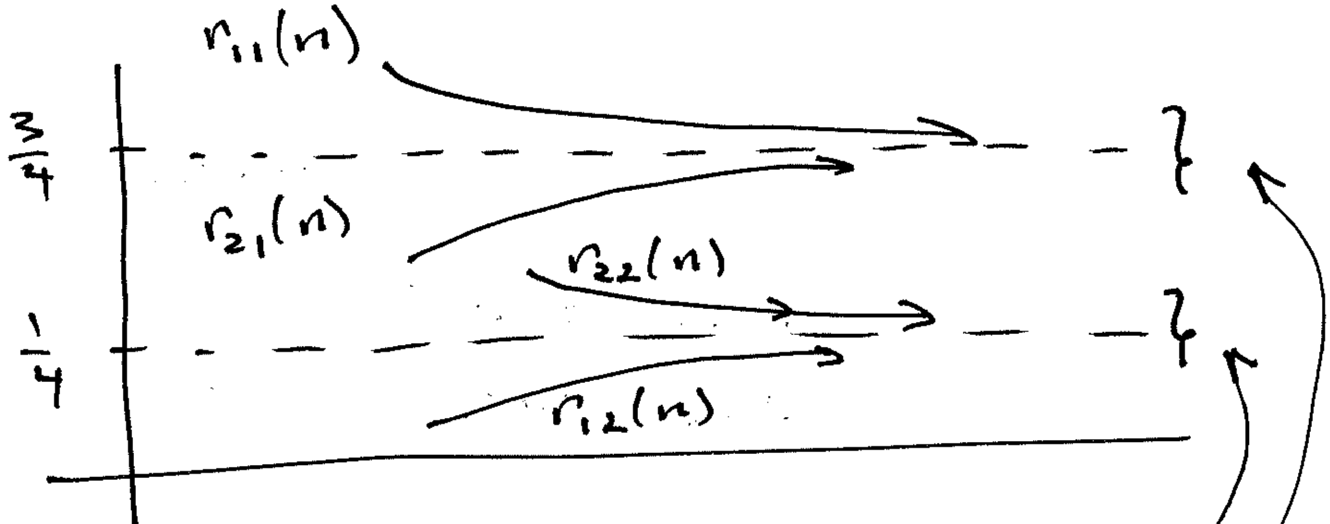
we can show (using L.A.)

$$\lim_{n\to\infty} R^{n} = \begin{pmatrix} .75 & .25 \\ .75 & .25 \end{pmatrix}$$

i.e.

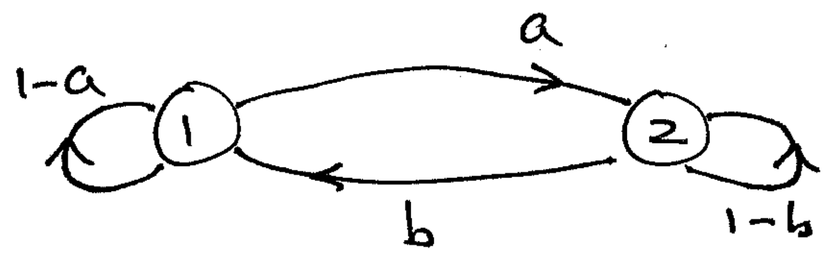
$$r_{11}(n) \longrightarrow \frac{3}{4}^{+}, \quad r_{12}(n) \longrightarrow \frac{1}{4}^{-}$$

$$r_{21}(n) \longrightarrow \frac{3}{4}^{-}, \quad r_{22}(n) \longrightarrow \frac{1}{4}^{+}$$



steady state Probabilities

Ex.

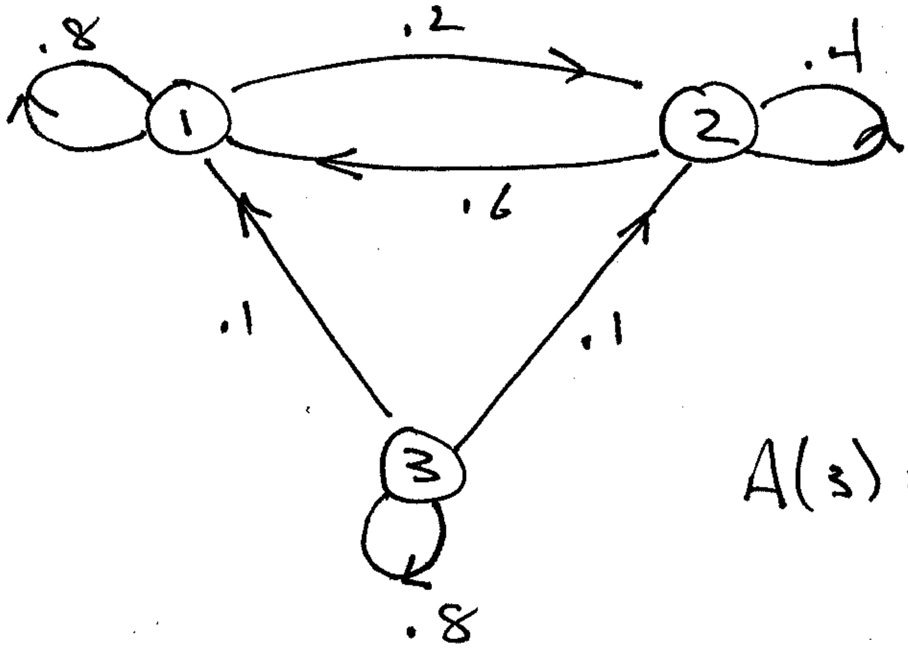


$$R = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

one can show

$$\lim_{n \to \infty} R^n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}$$

Ex. $A(1) = A(2) = \{1, 2\}$



$A(3) = \{1, 2, 3\}$

$$R = \begin{pmatrix} .8 & .2 & 0 \\ .6 & .4 & 0 \\ .1 & .1 & .8 \end{pmatrix}$$

one can show

$$\lim_{n \to \infty} R^n = \begin{pmatrix} .75 & .25 & 0 \\ .75 & .25 & 0 \\ .75 & .25 & 0 \end{pmatrix}$$

7.2 Classification of states

Defn

We say state j is <u>accessible</u> from state i (or that j is reachable from i) iff

$$r_{ij}(n) > 0 \quad \text{for some } n \geq 1$$

Equivalently, there exists

$$i, i_1, i_2, \ldots, i_{n-1}, j$$

such that the transitions

$$(i) \to (i_1) \to (i_2) \to \cdots \to (i_{n-1}) \to (j)$$

all have positive probability

notation: $i \dashrightarrow j$

means: j is accessible from i

Defn

Let $A(i)$ denote the set of states accessible from i

$$A(i) = \{ j \in S \mid i \dashrightarrow j \}$$

Defn

A state i is called <u>recurrent</u> iff for every $j \in A(i)$, we also have $i \in A(j)$

Equivalently:

$$j \in A(i) \text{ iff } i \in A(j)$$

i.e.

$$i \dashrightarrow j \quad \text{iff} \quad j \dashrightarrow i$$

Thus if we depart a recurrent state, there is a pos. prob. that we return. So after <u>long</u> enough time, we will return.

Thus a recurrent state, if it is visited at all, is visited infinitely often.