CSE 107 11-8-25

Supplemental Lecture

Exercise: Do same thing for Geometric(p).

Let Y = # flips to 1st head

Alice notices at time m that head has not occurred. Let

X = # flips after m^{th} to 1st head

$$\overbrace{\underbrace{0\,0\,0\cdots\cdots0}_{m}\;\underbrace{0\,0\cdots\cdots0\,0\,1}_{X}}^{Y}$$

let $A = \{Y > m\}$.

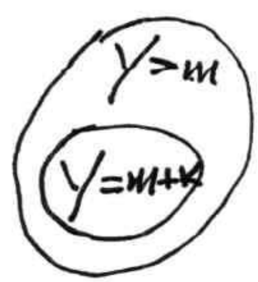
Find $p_{X|A}(k)$ for $k=1,2,3,\ldots$

so

$$p_{X|A}(k) = P(X=x|A)$$

$$= P(Y-m=k \mid Y>m)$$

$$= \frac{P(Y=m+k,\ Y>m)}{P(Y>m)}$$



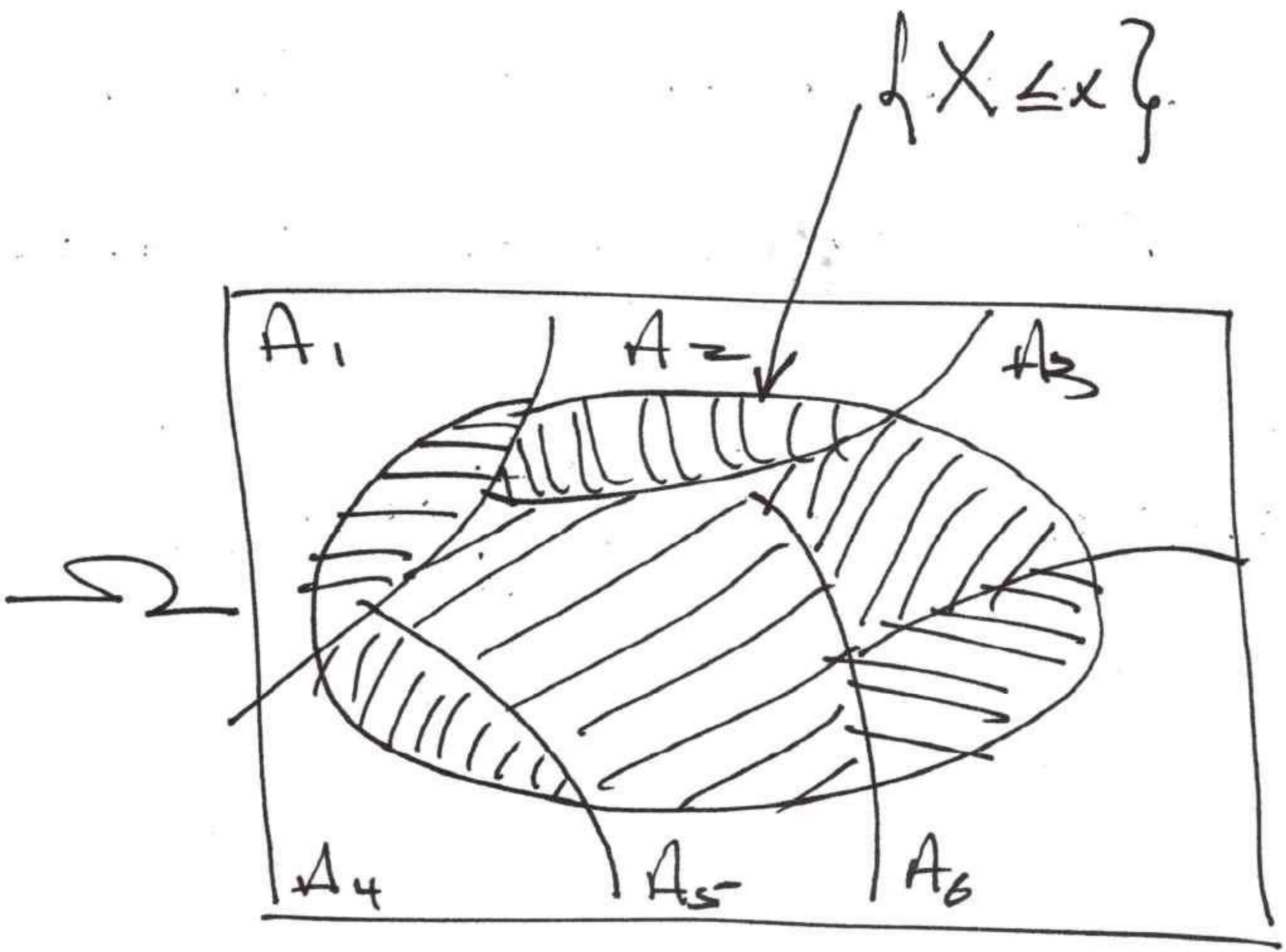
$$= \frac{P(Y=m+k)}{P(Y>m)}$$

$$= \frac{p(1-p)^{\cancel{m}+k-1}}{\cancel{(1-p)^{m}}}$$

$$= \boxed{p(1-p)^{k-1}} = p_Y(k) \quad (k=1,2,\ldots)$$

If events $A_1, A_2, \ldots, A_n$ form a Partition of Ω, then the total Probability thm gives

$$P(X \leq x) = \sum_{i=1}^{n} P(A_i) \cdot P(X \leq x \mid A_i)$$



so

$$F_X(x) = \sum_{i=1}^{n} P(A_i) \cdot F_{X|A_i}(x)$$

differentiate w.r.t. x to get

$$\boxed{f_X(x) = \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)}$$

Total Probability for densities.

Defn

if X, Y are jointly continuous r.v.s and $f_Y(y) > 0$ for all $y \in \mathbb{R}$, the Conditional PDF of X given Y

is

$$\boxed{f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

observe

$$\int_{-\infty}^{\infty} f_{X|Y}(x,y)dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)} dx$$

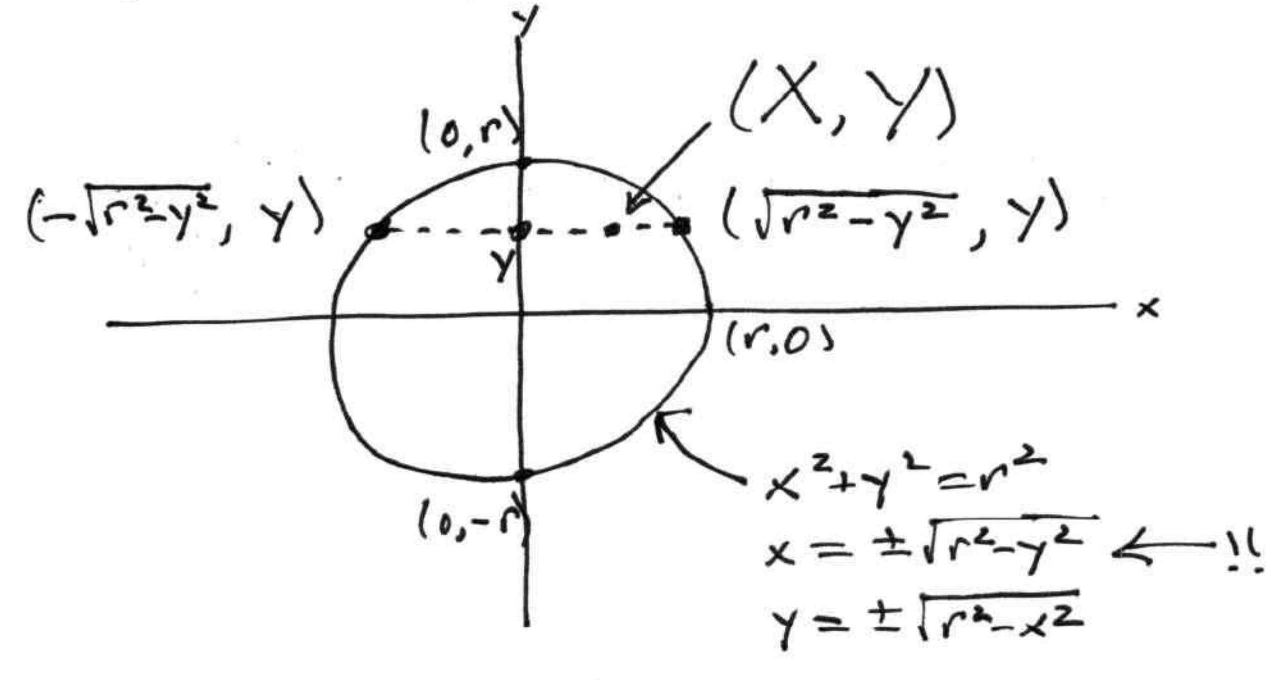
$$= \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx}{f_Y(y)} = \frac{f_Y(y)}{f_Y(y)} = 1$$

so $f_{X|Y}(x|y)$ is a valid PDF.

Ex

Throw a dart at a circular target of radius r, centered at $(0,0)$. Assume target is always hit, but otherwise impact point (x,y) is uniformly distributed. Let X, Y be the coordinates of the impact point. Find

✓ $f_{X,Y}(x,y)$, ✓ $f_Y(y)$ and ✓ $f_{X|Y}(x|y)$



we are given $f_{X,Y}(x,y)$ is uniform

so

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & \text{if } x^2+y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\,dx$$

$$= \frac{1}{\pi r^2} \cdot 2\sqrt{r^2-y^2}$$

$$= \begin{cases} \frac{2\sqrt{r^2-y^2}}{\pi r^2} & \text{if } |y| \le r \ (-r \le y \le r) \\ 0 & \text{otherwise} \end{cases}$$

<u>note</u>: although $f_{X,Y}(x,y)$ is uniform, $f_Y(y)$ is not.

Finally

$$f_{X|Y}(x|y) = \frac{f_{X|Y}(x,y)}{f_Y(y)}$$

$$= \begin{cases} \dfrac{1}{2\sqrt{r^2-y^2}} & \text{if } x^2+y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}$$

Thus, for fixed y, $f_{X|Y}(x|y)$ is a constant function of x.

All the same identities that hold for PMFs also hold for PDFs, like:

$$f_{X|Y}(x|y) \cdot f_Y(y) = f_{X,Y}(x,y)$$

$$= f_{Y|X}(y|x) \cdot f_X(x)$$

or multiplication rule:

$$f_{X,Y,Z}(x,y,z) = f_{X|Y,Z}(x|y,z)\, f_{Y|Z}(y|z) \cdot f_Z(z)$$

Conditional Expectation

Let X, Y be jointly continuous r.v.s and $A \subseteq \Omega$ with $P(A) > 0$

Defns

- $E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)\, dx$

- $E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx$

Expected Value rules

- $E[g(X)|A] = \int_{-\infty}^{\infty} g(x) \cdot f_{X|A}(x)\, dx$

- $E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\, dx$

Total Expectation Theorem

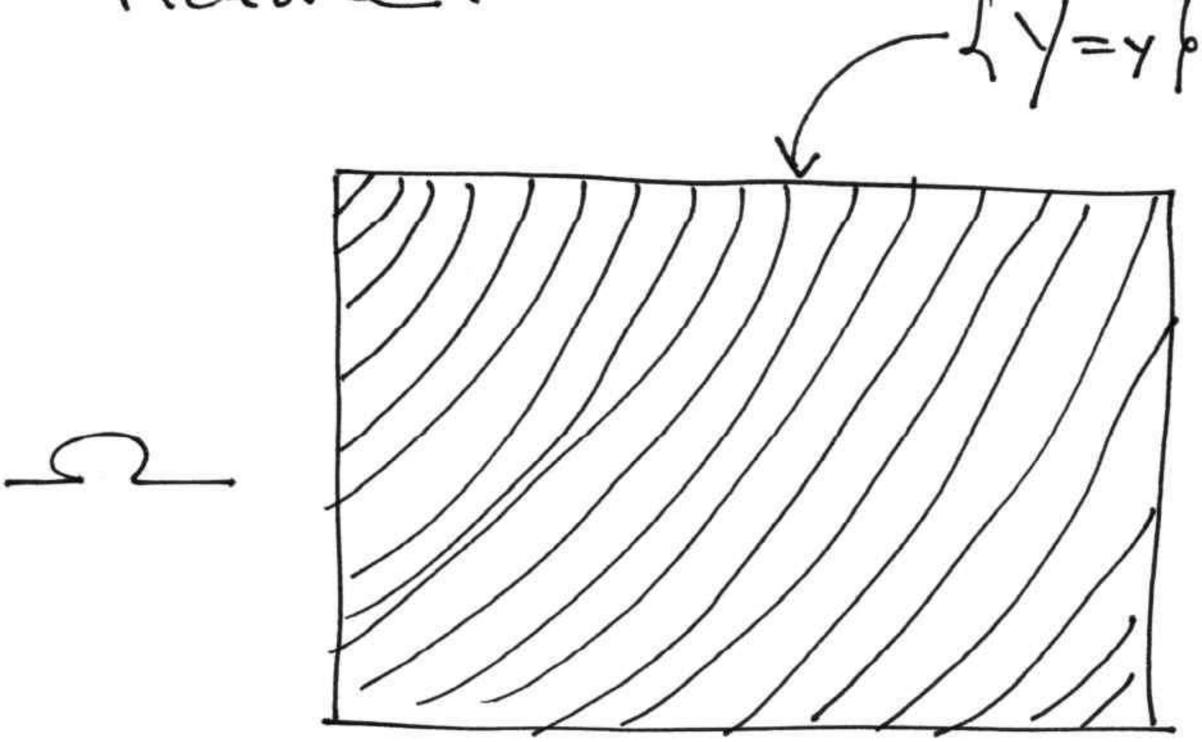
- If $A_1, \ldots, A_n$ form a partition of Ω and $P(A_i) > 0$ (all i), then

$$(1) \quad E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i]$$

- similarly, replace A_i by $A_y = \{Y = y\}$, for $y \in$ range(Y), then

$$(2) \quad E[X] = \int_{-\infty}^{\infty} f_Y(y) \cdot E[X \mid Y = y] \, dy$$

Picture:



Proof of (1)

start with tot. Prob. thm:

$$f_X(x) = \sum_{i=0}^{n} P(A_i) \cdot f_{X|A_i}(x)$$

mult. by x, integrate over $(-\infty,\infty)$:

$$\int_{-\infty}^{\infty} x f_X(x)\, dx = \sum_{i=1}^{n} P(A_i) \int_{-\infty}^{\infty} x f_{X|A_i}(x)\, dx$$

Thus

$$E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X|A_i]$$

Proving (1)

Proof of (2)

Start with RHS

$$\int_{-\infty}^{\infty} E[X|Y=y] \cdot f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx \right) \cdot f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X|Y}(x|y) \cdot f_Y(y)\; dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y}(x,y)\; dy\, dx$$

$$= E[X] \quad \left(\text{by notes 11-6-25 Page 1} \right)$$

$\blacksquare$

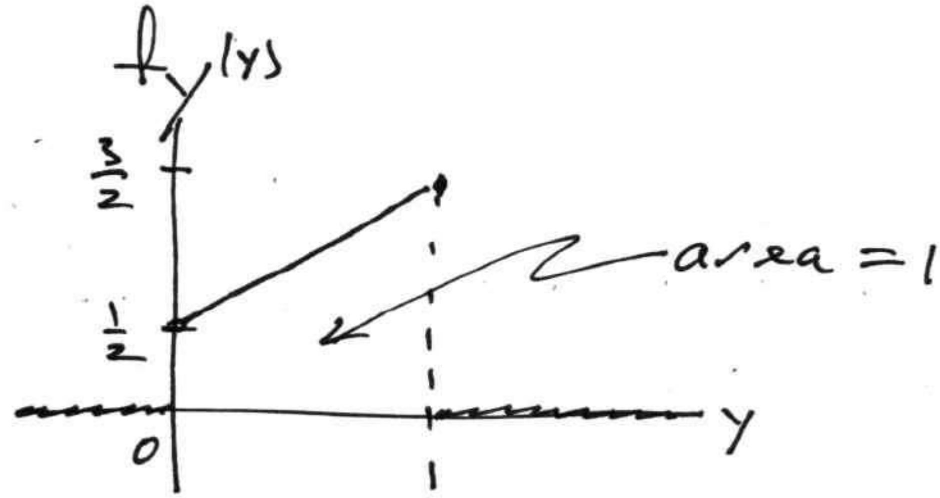
Ex

Let be jointly continuous with

$$f_Y(y) = \begin{cases} y + \frac{1}{2} & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{x+y}{y+\frac{1}{2}} & \text{if } 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



Find $E[X]$.

<u>solution</u>

$$E[X|Y=y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) dx$$

$$= \int_0^1 x\left(\frac{x+y}{y+\frac{1}{2}}\right) dx$$

$$= \frac{1}{y+\frac{1}{2}} \int_0^1 (x^2 + xy) dx$$

$$= \frac{1}{y+\frac{1}{2}} \cdot \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 y\right)\Big|_0^1$$

$$= \frac{\frac{1}{2}y + \frac{1}{3}}{y + \frac{1}{2}} \qquad (0 \le y \le 1)$$

and

$$E[X] = \int_{-\infty}^{\infty} E[X \mid Y=y] \cdot f_Y(y)\, dy$$

$$= \int_0^1 \left(\frac{\frac{1}{2}y + \frac{1}{3}}{\cancel{y+\frac{1}{2}}} \right) \cdot \cancel{(y+\tfrac{1}{2})}\, dy$$

$$= \frac{1}{2} \cdot \frac{1}{2} y^2 + \frac{1}{3} y \Big|_0^1 = \frac{1}{4} + \frac{1}{3} = \boxed{\frac{7}{12}}$$

<u>Another way</u>

$$f_{X,Y}(x,y) = f_{X \mid Y}(x \mid y) \cdot f_Y(y)$$

$$= \begin{cases} x+y & \text{if } 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$\to 0$

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_0^1 \int_0^1 x(x+y)\, dy\, dx$$

$$= \text{- - - - - - - -} = \frac{7}{12} \quad \text{(check)}$$

$\blacksquare$

Back up in this example

$$E[X \mid Y=y] = \begin{cases} \dfrac{\frac{1}{2}y + \frac{1}{3}}{y + \frac{1}{2}} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

let

$$g(y) = E[X \mid Y = y]$$

we can compose g with Y:

$$g(Y) = \underbrace{E[X \mid Y]}_{\text{this is a random variable}}$$

Previous example: $E[E[X|Y]] = \frac{7}{12}$

In general:

$$\boxed{E[E[X|Y]] = E[X]}$$

formula (2)   Law of Iterated Expectations