cse 107 11-6-25

observe ...

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right) dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\, dy\, dx$$

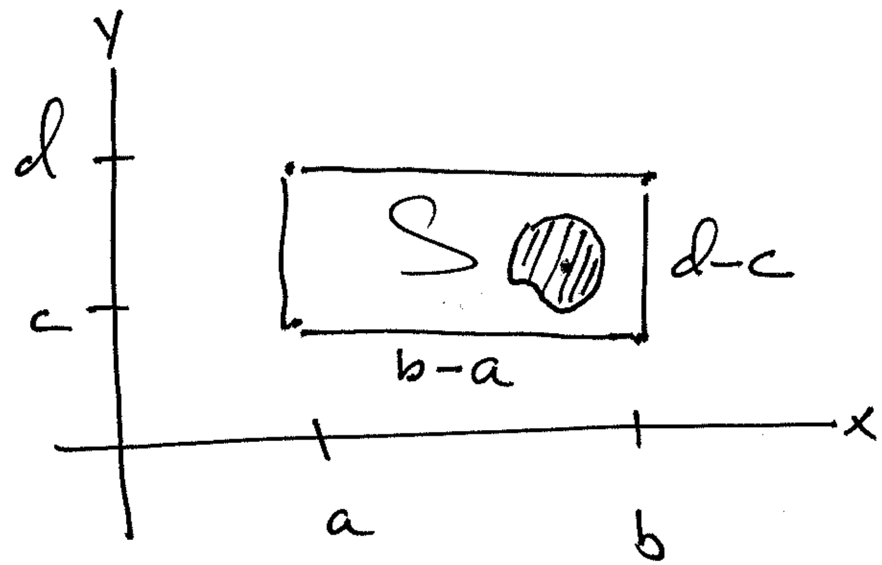
similarly

$$E[Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, f_{X,Y}(x,y)\, dx\, dy$$

Ex. Joint Uniform PDF on

$S = [a,b] \times [c,d]$

$$f_{X,Y}(x,y) = \begin{cases} \alpha & \text{if } (x,y) \in S \\ 0 & \text{otherwise} \end{cases}$$



We must have

$$1 = P(\Omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_c^d \int_a^b \alpha \, dx \, dy$$

$$= \alpha \int_c^d (b-a) \, dy = \alpha (b-a)(d-c)$$

$$= \alpha \cdot \text{area}(S)$$

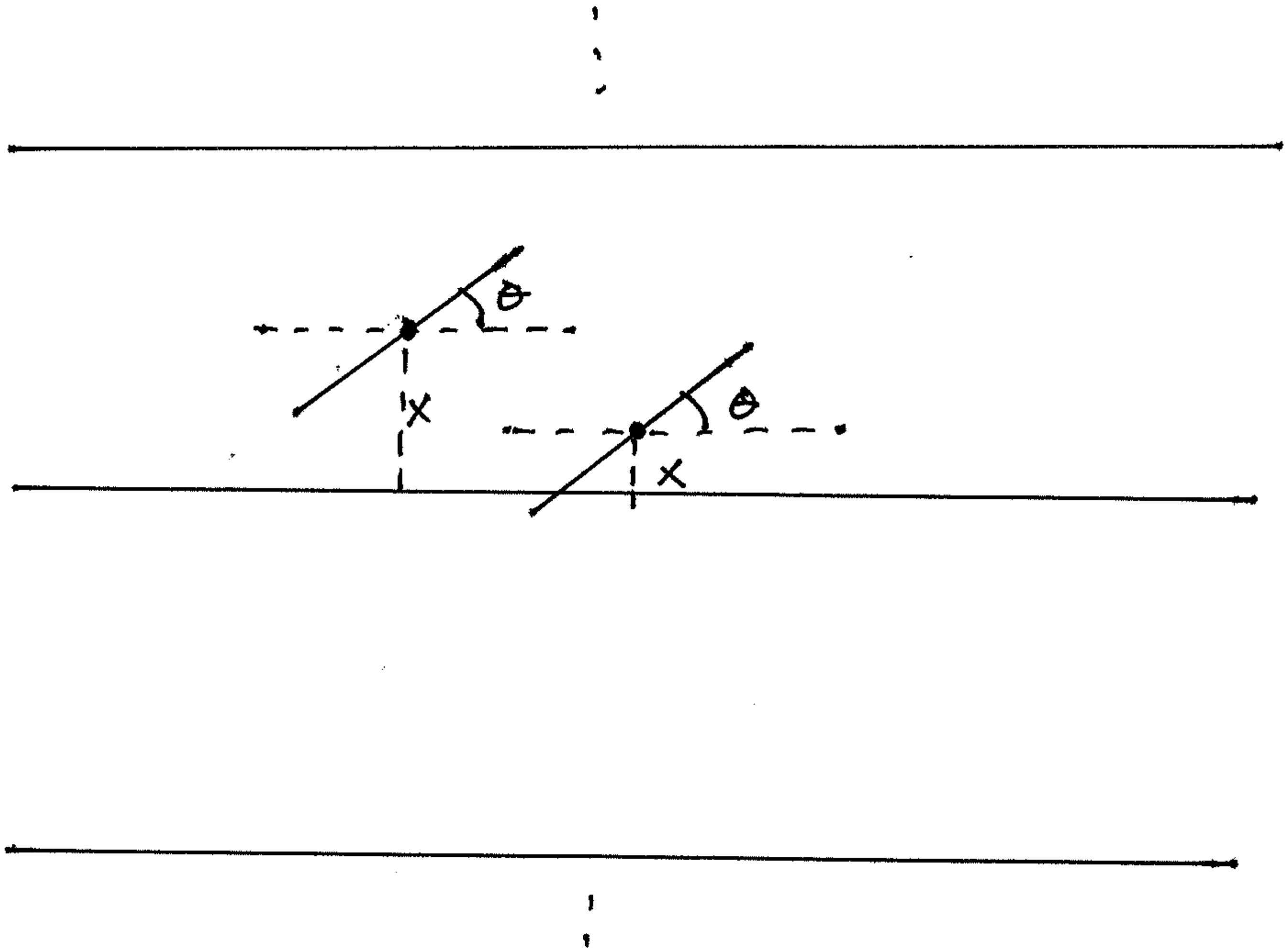
$$\therefore \quad \alpha = \frac{1}{\text{area}(S)}$$

$$\therefore \quad f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{area}(S)} & \text{if } (x,y) \in S \\ 0 & \text{otherwise} \end{cases}$$

Ex. Buffon's Needle

A surface is ruled with Parallel lines distance d apart. a needle of length ℓ is tossed on the surface. What is the probability that the needle intersects a line?

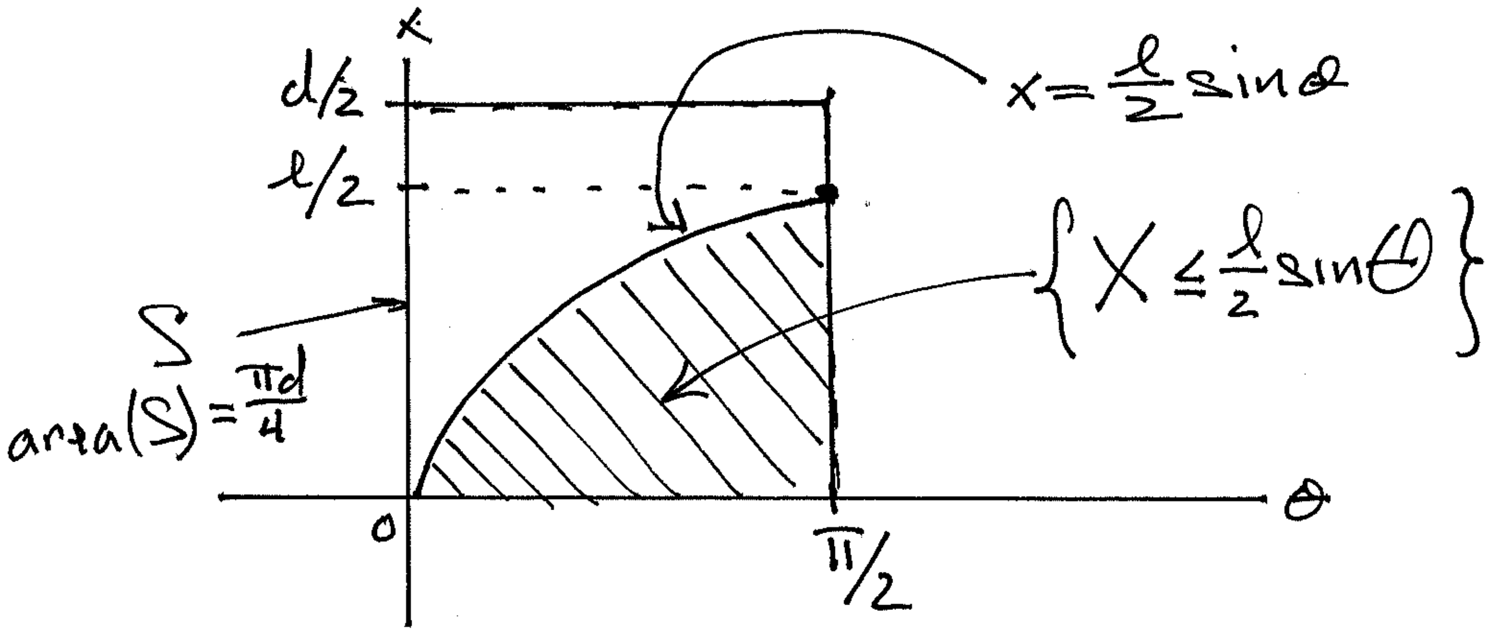
Assume $\ell < d$ so cannot intersect 2 lines.



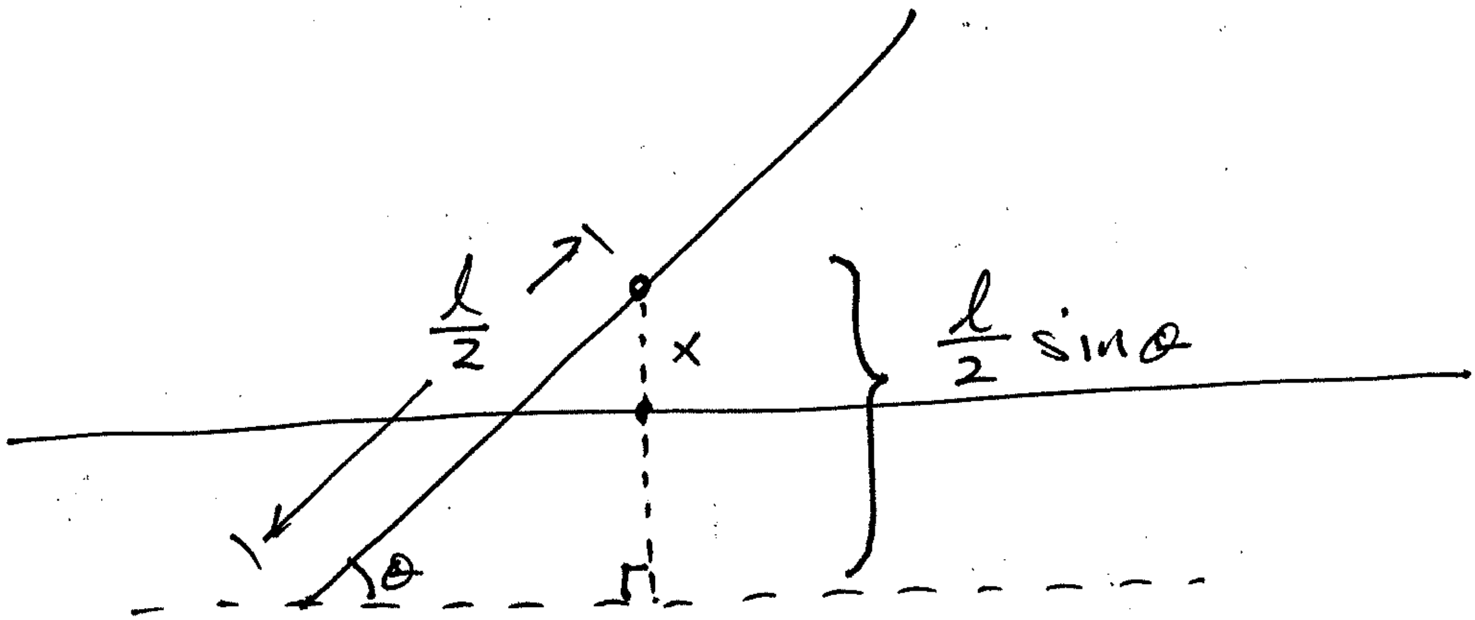
Let X be the vertical distance from the midpoint of the needle to the nearest line, and Θ the acute angle made by the needle and the horizontal.

We model the pair (X, Θ) with a joint uniform PDF.

note: $0 \le X \le \frac{\ell}{2}$ and $0 \le \Theta \le \frac{\pi}{2}$



$$f_{X,\Theta}(x,\theta) = \begin{cases} \dfrac{4}{\pi d} & \text{if } \begin{cases} 0 \le x \le \frac{d}{2} \\ \text{and} \\ 0 \le \theta \le \frac{\pi}{2} \end{cases} \\ 0 & \text{otherwise} \end{cases}$$



intersect $\Leftrightarrow$ $X \le \frac{l}{2}\sin\Theta$.

Thus

$$P\left(X \le \frac{l}{2}\sin\Theta\right) = \iint_{x \le \frac{l}{2}\sin\theta} f_{X,\Theta}(x,\theta)\,dx\,d\theta$$

$$= \frac{4}{\pi d}\int_0^{\pi/2}\int_0^{\frac{l}{2}\sin\theta} dx\,d\theta$$

$$= \frac{4}{\pi d} \int_0^{\pi/2} \frac{l}{2} \sin\theta \, d\theta$$

$$= \frac{2l}{\pi d} \left(-\cos\theta\right) \Big|_0^{\pi/2}$$

$$= -\frac{2l}{\pi d} \left(0 - 1 \right) = \frac{2l}{\pi d}$$

if $l = d$, then

$$\boxed{P(\text{intersect}) = \frac{2}{\pi}}$$

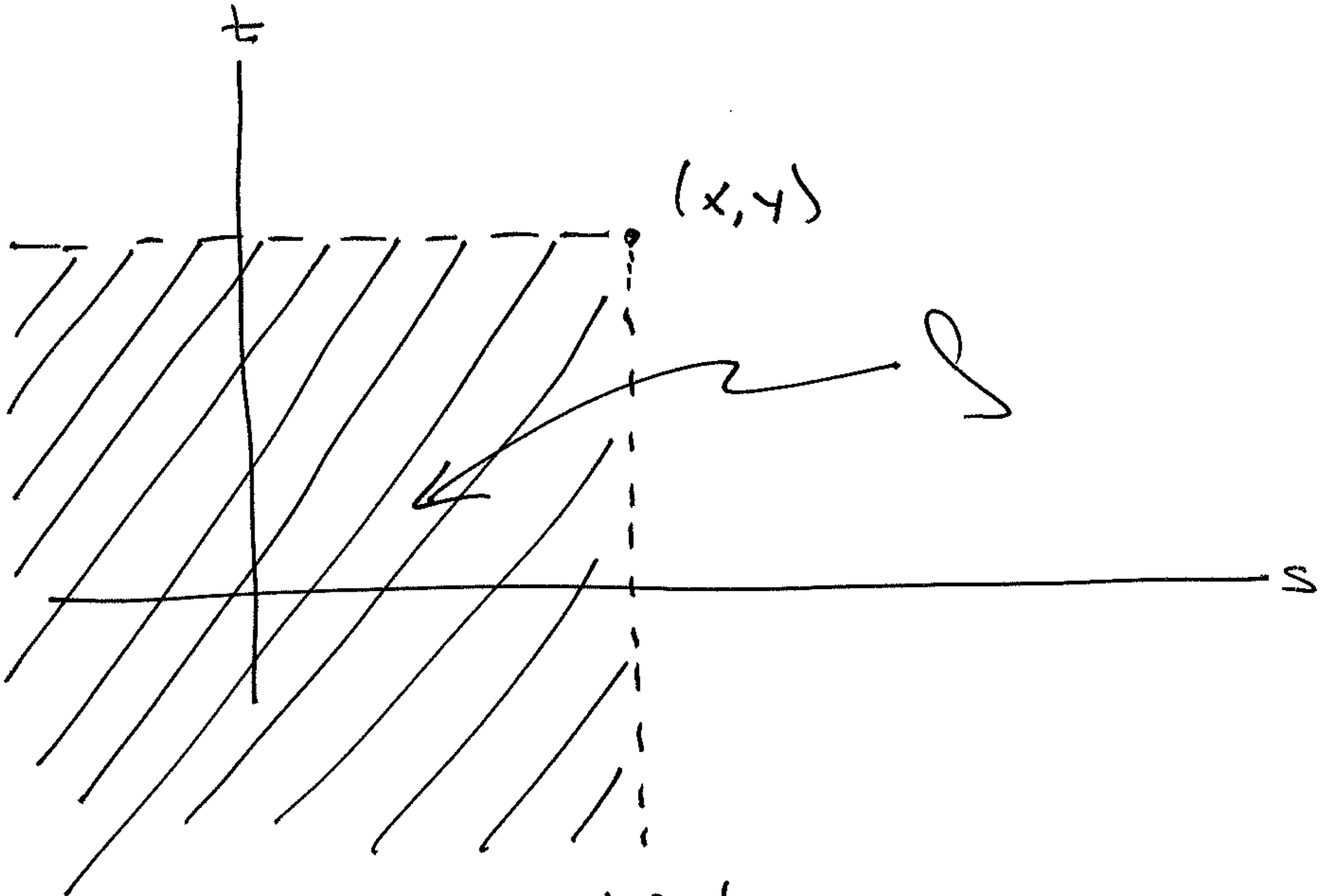
$$\therefore \quad \pi = \frac{2}{P(\text{intersect})}$$

Joint CDFs

if X, Y are 2 r.v.s, their joint CDF is

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\, dt\, ds$$



We can recover Joint PDF from Joint CDF by partial differentiation

$$\boxed{f_{X,Y}(x,y) = \frac{\partial^2 F}{\partial y\, \partial x}(x,y)}$$

Expectation

if $g : \mathbb{R}^2 \to \mathbb{R}$ we have

$$E[g(X,Y)] = \iint_{\mathbb{R}^2} g(x,y) \cdot f_{X,Y}(x,y)\, dx\, dy$$

Special case

$$E[aX + bY + c] = \text{.... exercise ....}$$

$$= aE[X] + bE[Y] + c$$

Same for 3 r.v.s ...

$$E[aX + bY + cZ + d]$$

$$= aE[X] + bE[Y] + cE[Z] + d$$

3.5 Conditioning

Let A be an event, with $P(A) > 0$, and X a continuous r.v.

Defn

The <u>conditional PDF of X given A</u> is a function $f_{X|A}(x)$ satisfying

$$P(X \in I \mid A) = \int_I f_{X|A}(x)\,dx$$

for 'any' subset $I \subseteq \mathbb{R}$.

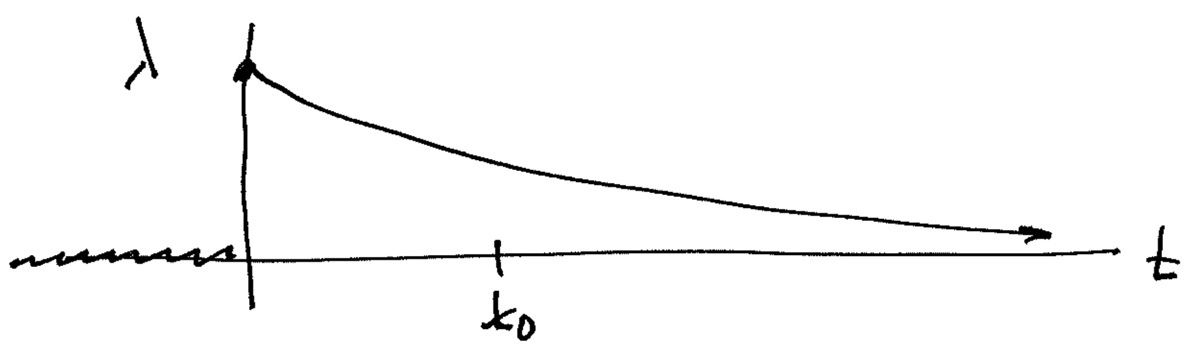
We require

$$\int_{-\infty}^{\infty} f_{X|A}(x)\,dx = 1$$

Ex.

The time T until a new light bulb burns out is Exponential(λ), i.e.

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Alice turns on the light, leaves and returns t_0 hours later the light still on. Thus the event

$$A = \{T > t_0\}$$

has occurred. Let X be the additional time to burnout. Thus

$$X = T - t_0$$

$$T = X + t_0$$

find the conditional PDF

$$f_{X|A}(x)$$

<u>solution</u>

Plan: 1st find conditional CDF

$$F_{X|A}(x) = P(X \le x \mid A)$$

$$= \int_{-\infty}^{x} f_{X|A}(s)\, ds$$

then differentiate:

$$f_{X|A}(x) = \frac{d}{dx} F_{X|A}(x)$$

Recall

$$P(T > a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \\ 1 & \text{if } a \leq 0 \end{cases}$$

Thus

$$P(X > x \mid A) = P(T > t_0 + x \mid T > t_0)$$

$$= \frac{P(T > t_0 + x,\ T > t_0)}{P(T > t_0)}$$

$$= \frac{P(T > t_0 + x)}{P(T > t_0)}$$

$$= \frac{e^{-\lambda(t_0+x)}}{e^{-\lambda t_0}}$$

$$= \frac{\cancel{e^{-\lambda t_0}} \cdot e^{-\lambda x}}{\cancel{e^{-\lambda t_0}}} = e^{-\lambda x}$$

$$\therefore F_{X|A}(x) = P(X \le x \mid A)$$

$$= 1 - P(X > x \mid A)$$

$$= \begin{cases} 1 - e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Hence

$$f_{X|A}(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Which is Exponential(λ).

<u>Remark</u>

This shows that the exponential r.v. is memoryless !!