CSE 107 11-4-25

<u>Theorem</u>

If X is normal(μ, σ), and $a \neq 0$, b are constants, then

$$Y = aX + b$$

is also <u>normal</u> with mean

$$E[Y] = a\mu + b$$

and variance

$$Var(Y) = a^2\sigma^2.$$

<u>note</u>: The significant Part of this theorm is that $Y = aX + b$ is normal.

<u>Proof</u>

let $Y = aX + b$ (assume $a > 0$).

Then

$$F_Y(y) = P(Y \le y)$$

$$= P(aX + b \le y)$$

$$= P\left(X \le \frac{y-b}{a}\right)$$

$$\therefore \quad F_Y(y) = F_X\left(\frac{y-b}{a}\right)$$

$$\frac{d}{dy} \downarrow$$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{\frac{-\left(\frac{y-b}{a} - \mu\right)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\,(a\sigma)} \cdot e^{\frac{-\left(\frac{y-b-a\mu}{a}\right)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi} \cdot (a\sigma)} \cdot e^{\frac{-(y-(a\mu+b))^2}{2(a\sigma)^2}}$$

This the PDF of the r.v. Normal$(a\mu+b, a\sigma)$

<u>Defn</u>

The Standard Normal r.v. is the normal with $\mu=0$ and $\sigma=1$.

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}$$

Notation for std normal CDF

$$\Phi(y) = P(Y \le y)$$

$$= P(Y < y)$$

$$= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \cdot e^{-t^2/2} \, dt$$

Handout: standard normal table.

table gives $\Phi(y)$ for

$$0 \le y \le 3.49$$

if $y > 3.49$, approx. $\Phi(y)$ by 1.

if $y < 0$, use identity

$$\Phi(-y) = 1 - \Phi(y) \quad (y \in \mathbb{R})$$

<u>Ex</u>

- $\Phi(1.84) = .9671$
- $\Phi(-1.84) = 1 - \Phi(1.84) = .0329$

Let X be Normal(μ, σ), and let $\boxed{Y = \frac{X-\mu}{\sigma}}$. Then

Y is normal with

$$E[Y] = E\left[\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma}\cdot\mu - \frac{\mu}{\sigma} = 0 \quad ✓$$

and

$$\mathrm{Var}(Y) = \mathrm{Var}\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma^2}\mathrm{Var}(X)$$

$$= \frac{1}{\cancel{\sigma^2}}\cdot\cancel{\sigma^2} = 1 \quad ✓$$

so Y is std. normal.

Thus

$$F_X(x) = P(X \le x)$$

$$= P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$$

$$= P\left(Y \le \frac{x-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Ex.

The mean high temp in Santa Cruz on Nov. 1 is 66° F, with std. dev. 4.4° F. What is the prob. that the high temp will be $\leq 57^\circ$?

soln

let X be the high temp.

Then

$$P(X \leq 57) = P\left(\frac{X-66}{4.4} \leq \frac{57-66}{4.4}\right)$$

$$= P(Y \leq -2.04)$$

$$= \Phi(-2.04)$$

$$= 1 - \Phi(2.04)$$

$$= 1 - (.9793)$$

$$= \boxed{.0207}$$

Ex.

what is Prob. that a normal r.v. X is within 1 std. dev. of its mean.

Soln

$$P(\mu-\sigma \le X \le \mu+\sigma)$$

$$= P\left(-1 \le \frac{X-\mu}{\sigma} \le 1\right)$$

$$= P(-1 \le Y \le 1)$$

$\vdots$

$$= P(Y \le 1) - P(Y < -1)$$

$$= \Phi(1) - \Phi(-1)$$

$$= \Phi(1) - (1 - \Phi(1))$$

$$= 2\Phi(1) - 1$$

$$= 2(.8413) - 1 = \boxed{.6826}$$

Similarly (exercise)

$P(X$ within 2 s.d of mean$)$

$$= 2\Phi(2) - 1 = \boxed{.9544}$$

and

$$P(X \text{ is within 3 s.d. of mean})$$

$$= 2\,\Phi(3) - 1 = \boxed{.9974}$$

3.4 Joint PDFs, multiple r.v.s

Defn

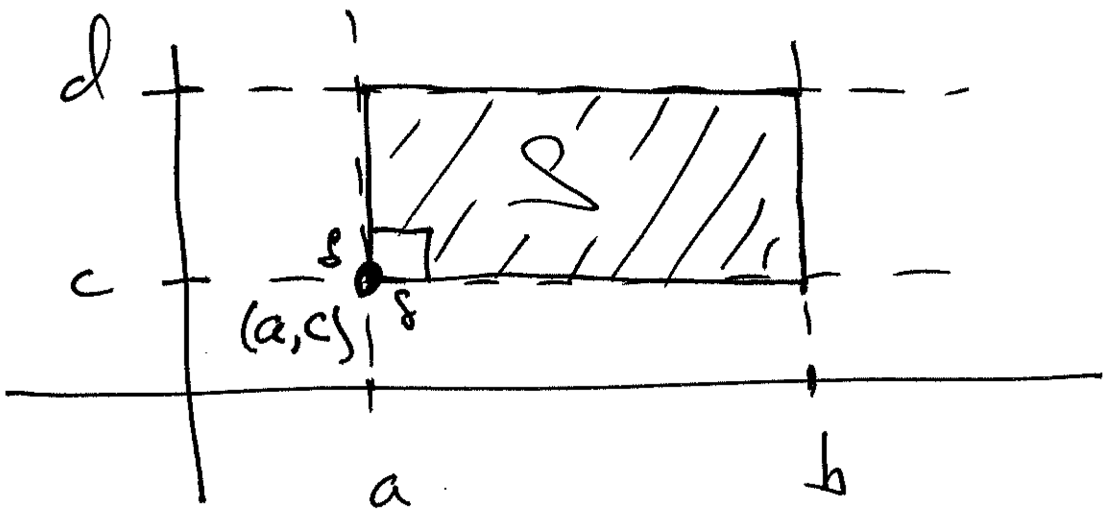
We say X, Y are jointly continuous iff there exists a function $f_{X,Y}(x,y)$ such that

$$P((X,Y) \in S) = \iint_S f_{X,Y}(x,y)\,dx\,dy$$

for 'any' subset $S \subseteq \mathbb{R}^2$.

In Particular if

$$S = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$$



then

$$P(a \le X \le b,\ c \le Y \le d)$$

$$= \int_a^b \int_c^d f_{X,Y}(x,y)\, dy\, dx$$

note if $S = \mathbb{R}$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = P(\Omega) = 1$$

If $\delta > 0$ is small, and $f_{X,Y}(x,y)$ is continuous near (a,c), then

$$P(a \le X \le a+\delta,\ c \le Y \le c+\delta)$$

$$= \int_{a}^{a+\delta}\int_{c}^{c+\delta} f_{X,Y}(x,y)\,dy\,dx$$

$$\approx f_{X,Y}(a,c) \cdot \delta^2$$

so $f_{X,Y}(a,c)$ has units

mass per unit area, i.e.

$f_{X,Y}(x,y)$ is probability density

at (x,y).

Recall if $I \subseteq \mathbb{R}$, we have

$$P(X \in I) = \int_I f_X(x)\,dx.$$

But also

$$P(X \in I) = P(X \in I, Y \in \mathbb{R})$$

$$= \iint_{I \times \mathbb{R}} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_I \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right) dx$$

Thus

$$\boxed{f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy}$$

Similarly

$$\boxed{f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}$$