CSE 107 11-28-25

Supplemental Lecture

Y_k = time to k^{th} arrival in a Bernoulli Process.

<u>Goal</u> Find $P_{Y_k}(t)$ for $t = k, k+1, \dots$

$$P_{Y_k}(t) = P(Y_k = t)$$

$$= P\{\text{have } (k-1) \text{ arrivals in } \{1,2,\dots,t-1\}, \text{ and } 1 \text{ arrival at time } t\}$$

$$= \underbrace{P((k-1) \text{ arrivals in } [1, t-1])}_{\text{Binomial}(p,\, t-1;\, k-1)} \cdot \underbrace{P(\text{arrival at } t)}_{p}$$

Thus

$$P_{Y_K}(t) = \binom{t-1}{K-1} p^{K-1} (1-p)^{(t-1)-(K-1)} \cdot p$$

$$\therefore \ P_{Y_K}(t) = \begin{cases} \binom{t-1}{K-1} p^K (1-p)^{t-K} & \text{if } t = K, K+1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

This is Pascal of order K

We have

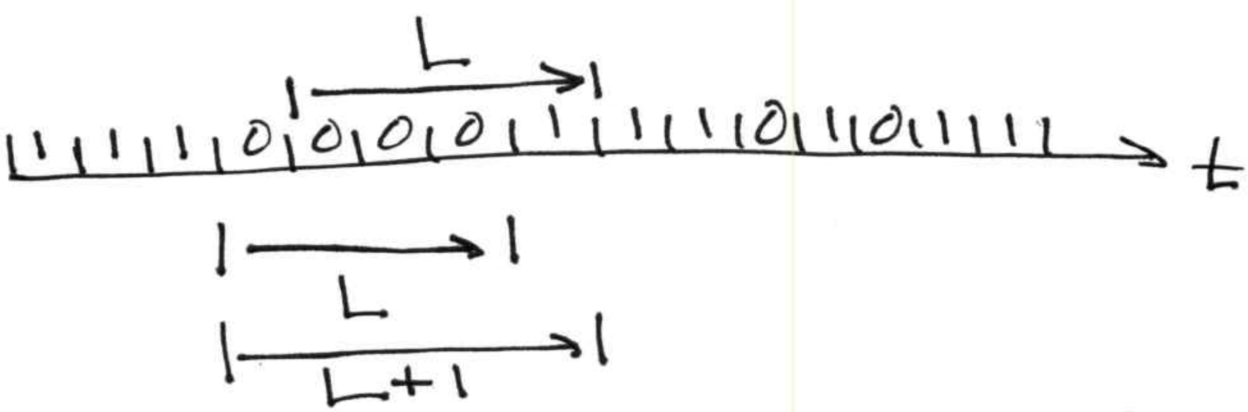
$$E[Y_K] = \frac{K}{p}, \quad Var(Y_K) = \frac{K(1-p)}{p^2}$$

Exercise check that

$$\sum_{t=K}^{\infty} P_{Y_K}(t) = 1 .$$

Ex.

Buy a lottery ticket everyday. Let L be the length of your 1^{st} losing streak.



what is the distribution of L ?

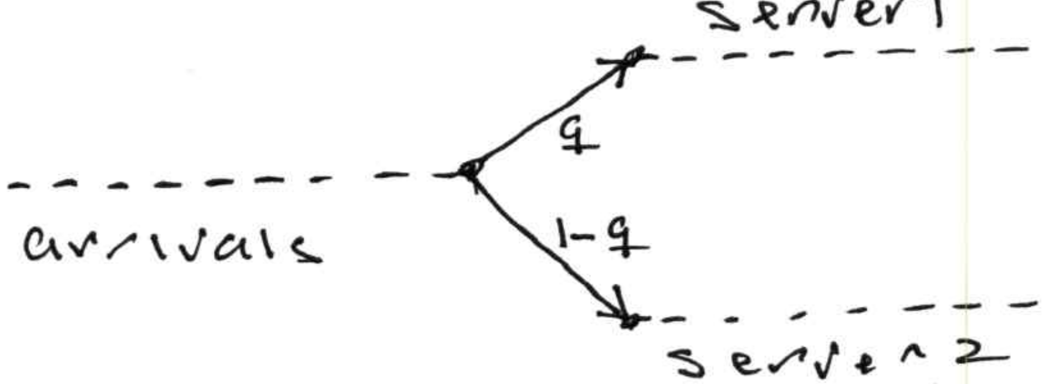
note: $L+1$ is not geometric, since L cannot be 0, hence $L+1$ cannot be 1.

L is time from 2^{nd} loss to next win, and is Geometric(p).

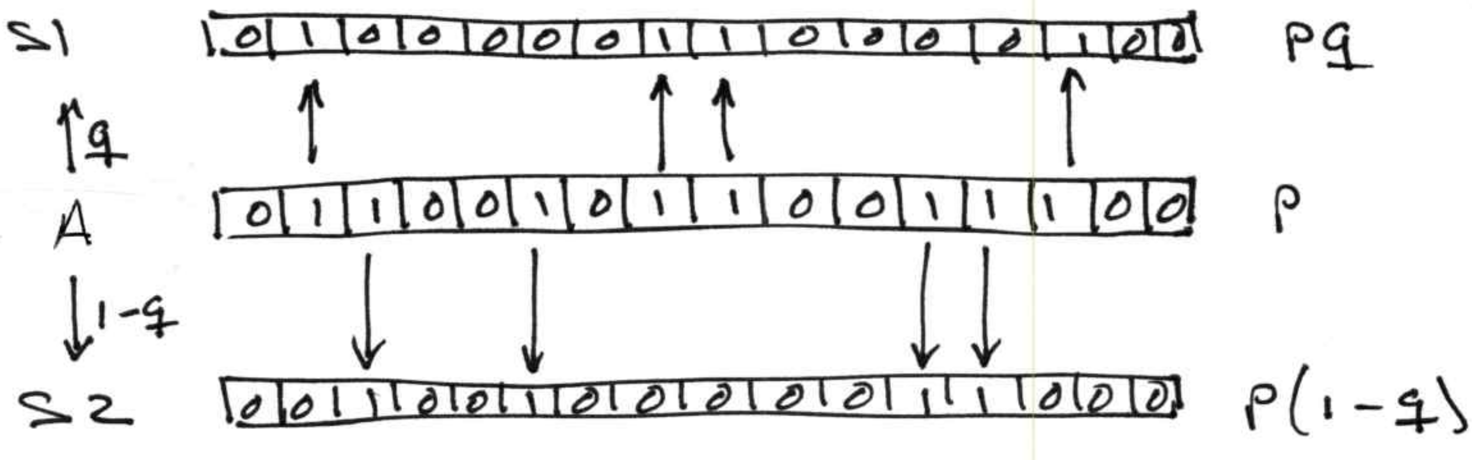
Splitting a Bernoulli Process

We have a Bernoulli Process feeding a stream of arrivals into a queue. Each job goes to one of 2 servers with

$$P(\text{server1}) = q$$
$$P(\text{server2}) = 1-q$$



What kind of arrival Processes are seen by S1 and S2?



$P(\text{arrival in } S1) = p \cdot q$

$P(\text{arrival in } S2) = p \cdot (1-q)$

Both S1, S2 Are Bernoulli Processes.

Merging 2 Bernoulli Processes

Given independent simultaneous Bernoulli Processes.

Parameter

S1 : $X_1, X_2, X_3 \ldots$ p

S2 : $Y_1, Y_2, Y_3 \ldots$ q

Merge events of S1 and S2 into a new Process

M : $M_1, M_2, M_3, \ldots$ $p+q-pq$

according to the rule

$$M_t = \begin{cases} 1 & \text{if either } X_t = 1 \text{ or } Y_t = 1 \\ 0 & \text{otherwise} \end{cases}$$

inclusive or

what Kind of arrival Processes is M ?

observe

$$P(M_t = 1) = 1 - P(M_t = 0)$$

$$= 1 - P(X_t = 0 \text{ and } Y_t = 0)$$

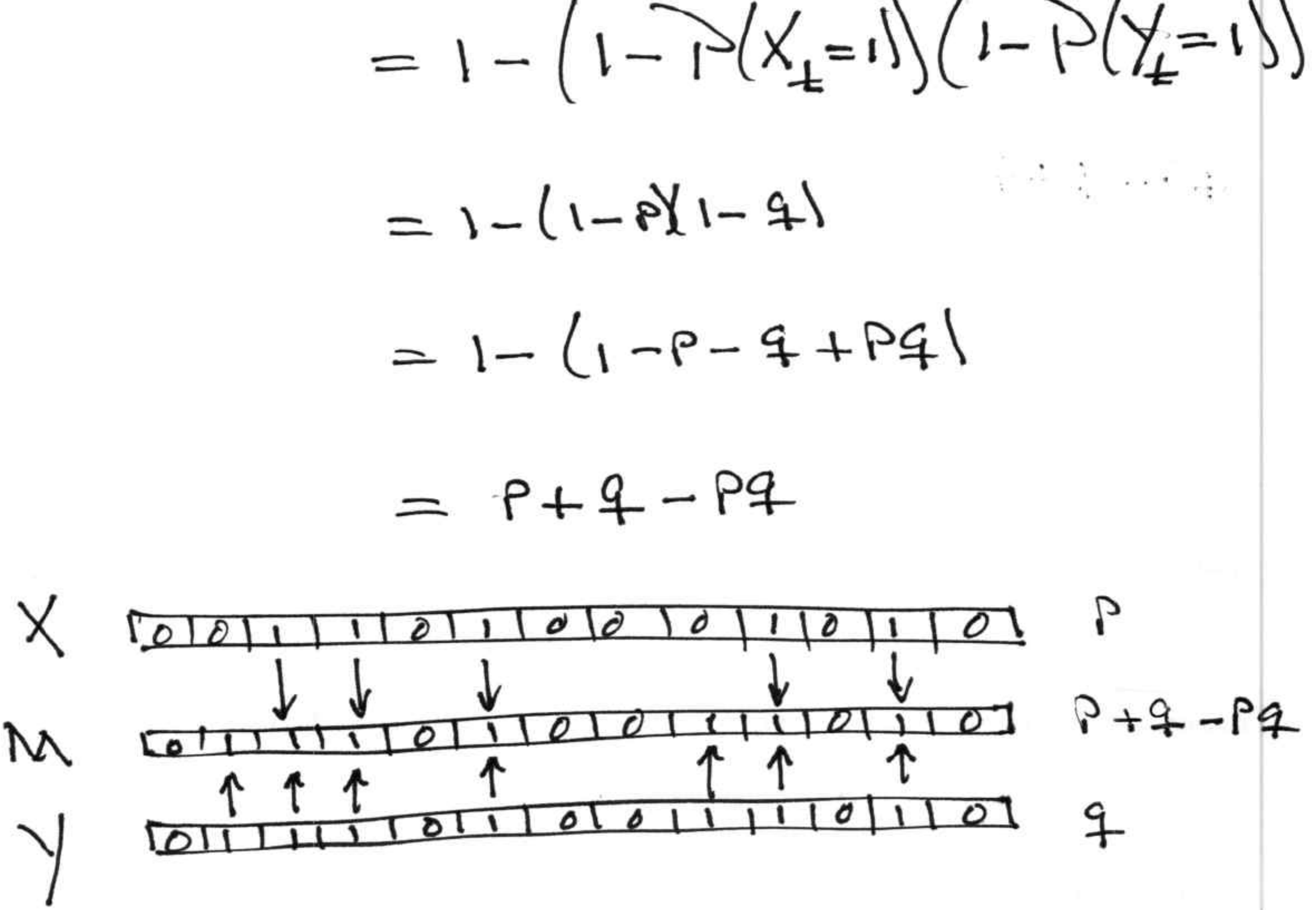
$$= 1 - P(X_t = 0) \cdot P(Y_t = 0)$$

$$= 1 - \big(1 - P(X_t = 1)\big)\big(1 - P(Y_t = 1)\big)$$

$$= 1 - (1-p)(1-q)$$

$$= 1 - (1 - p - q + pq)$$

$$= p + q - pq$$



M is a Bernoulli Process
with Parameter $p+q-pq$.

Exercise

- 1st split, then merge. Determine Parameter of resulting B.P.
- 1st merge, then split. Determine Parameters of resulting B.P.s
- change operation in Merge from <u>or</u> to ! and, exor, nand, nor ....
- split into multiple (say 3 or more) new Processes.

6.2 The Poisson Process

Recall:

$$X_1, X_2, \ldots.$$

is a Bernoulli Process iff

(1) each X_t is Bernoulli(p)

(2) $\{X_1, X_2, \ldots\}$ is independent.

i.e. events occuring in disjoint time intervals are independent.

We defined

- Y_k = time to k^{th} arrival

- $\begin{cases} T_1 = Y_k \\ T_k = Y_k - Y_{k-1} \end{cases}$ the interarrival times can also define

- $N_t = X_1 + X_2 + \dots + X_t \quad (t = 1, 2, 3, \dots)$

N_t = # of arrivals up to time t

observe N_t is Binomial (p, t), so

$$P_{N_t}(k) = \begin{cases} \binom{t}{k} p^k (1-p)^{t-k} & (k = 0, 1, 2, \dots) \\ 0 & \text{otherwise} \end{cases}$$

we can recover X_t from seq. of N_t by

$$\begin{cases} X_1 = N_1 \\ X_t = N_t - N_{t-1} & (t = 2, 3, \dots) \end{cases}$$

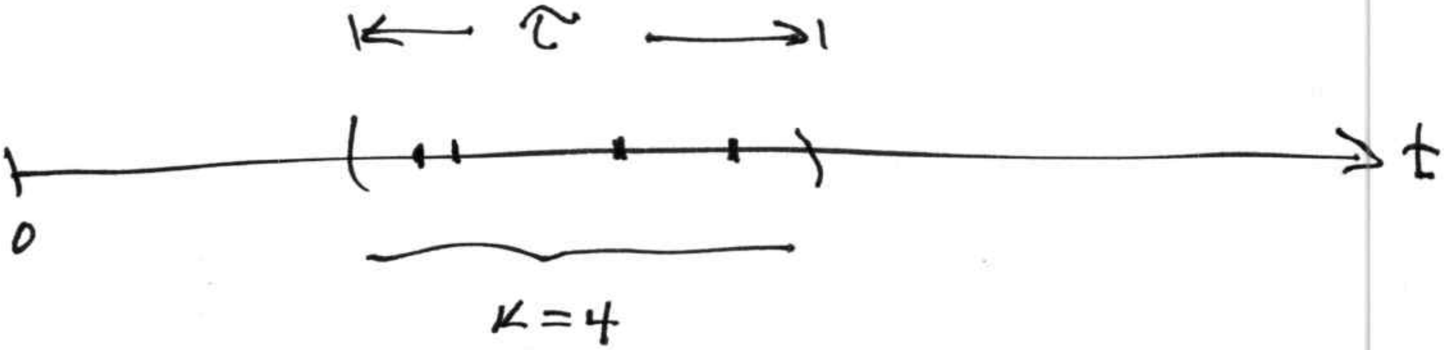
Thus any one of X_t, N_t, Y_k, T_k can be used to define a Bernoulli Process.

Poisson Processes

A continuous time arrival process is a discrete subset of $\mathbb{R}^+ = [0, \infty)$, i.e. the 'arrival times'.

We define

$$P(k, \tau) = P(\text{\# arrivals during an interval of length } \tau \text{ is } k)$$



Defn

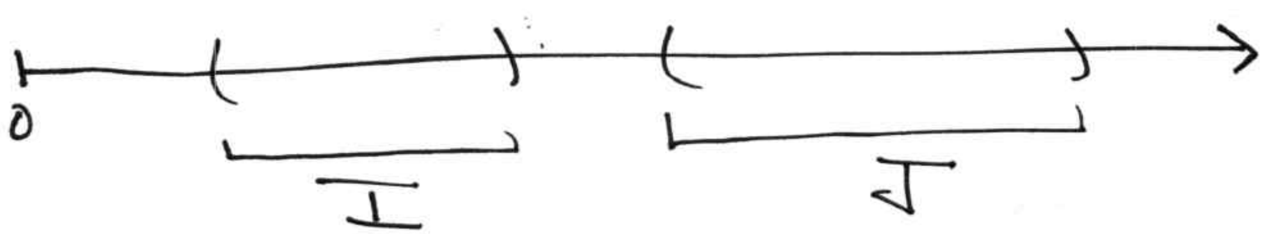
a continuous-time arrival Process is called a **Poisson Process** with rate $\lambda > 0$ iff

(a) **Time-Homogeneity**

$P(k, \tau)$ is same for all intervals of length τ.

(b) **Independence**

The numbers of arrivals in disjoint time intervals, are independent.



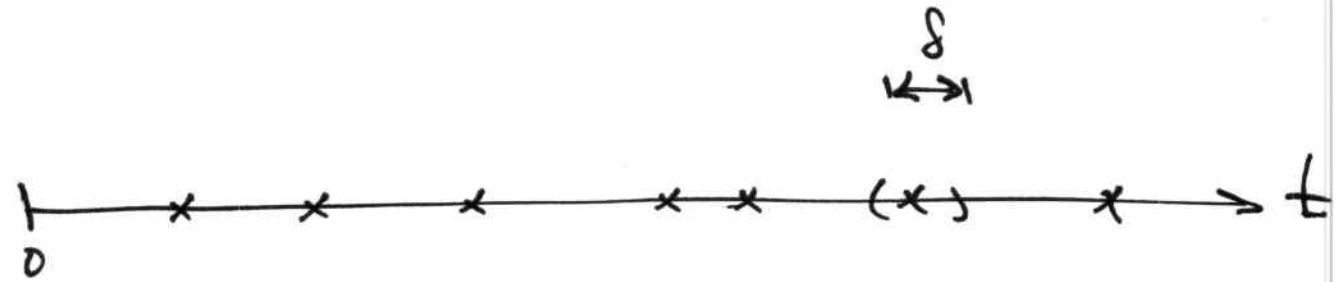
$I \cap J = \phi \Rightarrow$ # arrivals in I indep. of # arrivals in J.

(c) small interval Probabilities

let $\delta > 0$ be small. Then

$$P(k,\delta) \approx \begin{cases} 1-\lambda\delta & \text{if } k=0 \\ \lambda\delta & \text{if } k=1 \\ 0 & \text{if } k=2,3,\ldots \end{cases}$$

Here λ is the arrival rate.



note: $\approx$ in (c) means as $\delta \to 0$

$$P(k,\delta) = \left\{ \begin{matrix} 1-\lambda\delta & k=0 \\ \lambda\delta & k=1 \\ 0 & k\geq 2 \end{matrix} \right\} + o(\delta)$$

where $o(\delta)$ denotes a function with the property

$$\lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0$$

For instance, the function $c\delta^2$ has this property.

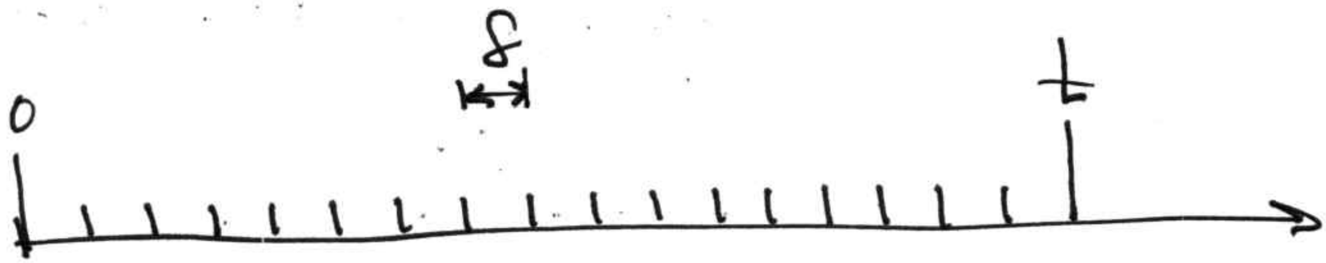
<u>observe</u>:

$$E[\# \text{ of arrivals in } [t, t+\delta]]$$

$$\approx (1-\lambda\delta)\cdot 0 + \lambda\delta \cdot 1 = \lambda\delta$$

$\therefore$ λ is the expected # of arrivals per unit time.

what about a large time interval?



let $n = \frac{t}{\delta}$ = # of subintervals.

choose δ so small that the Probability of more than 1 arrival is negligible.

we now have, approximately, a Bernoulli Process, with Probability of arrival,

$$p = \lambda\delta = \frac{\lambda t}{n}.$$

The approximation improves as $\delta \to 0$.

We know the # of successes (i.e. arrivals) in n trials of a Bernoulli Process with Parameters n, p

is Binomial (n, p), so approximately

$$P(k \text{ arrivals in } [0,t])$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k}$$

In the limit as $n \to \infty$ (so that $\delta \to 0$) this becomes Poisson with parameter λt. Thus

$$P(k \text{ arrivals in } [0,t])$$

$$= e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}$$

Let N_t = # of arrivals of a Poisson Process of rate λ in $[0, t]$. Then

$$P_{N_t}(k) = \begin{cases} e^{-\lambda t} \cdot \dfrac{(\lambda t)^k}{k!} & k = 0, 1, 2 \ldots \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$E[N_t] = \lambda t$$

$$Var(N_t) = \lambda t$$

Let T be time to 1st arrival in a Poisson Process

Goal: find distribution of T.

observe $\{T > t\} = \{N_t = 0\}$. Thus

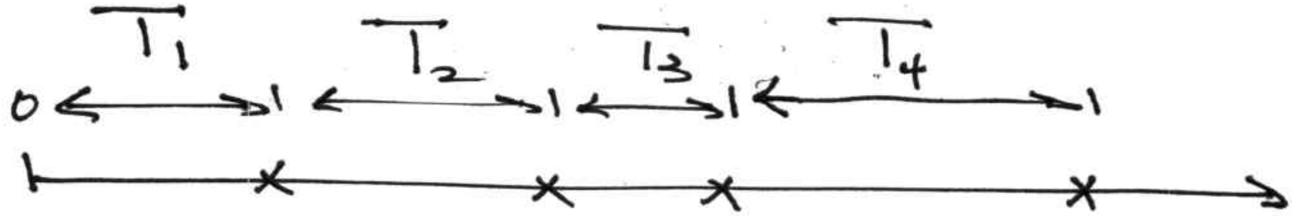
$$
\begin{aligned}
F_T(t) &= P(T \le t) \\
&= 1 - P(T > t) \\
&= 1 - P(N_t = 0) \\
&= 1 - p_{N_t}(0) \\
&= 1 - e^{-\lambda t}
\end{aligned}
$$

Differentiate w.r.t. t to get

$$f_T(t) = \lambda e^{-\lambda t} \quad \text{for } t \geq 0.$$

$\therefore$ T is exponential(λ).

Now let T_k be k^{th} interarrival time $(k = 1, 2, 3, \ldots)$



After each arrival, the Poisson Process has a 'fresh start'. Hence each T_k is also Exponential(λ)

$$f_{T_k}(t) = \lambda e^{-\lambda t} \quad \text{for } t \geq 0$$

Now define

$$Y_K = T_1 + T_2 + \cdots + T_K \quad (K = 1, 2, \ldots)$$

The time to the K^{th} arrival.

see text p. 316–317 for distribution of Y_K.

Erlang distribution of order K

$$f_{Y_K}(t) = \frac{\lambda^K t^{K-1} e^{-\lambda t}}{(K-1)!} \quad (t \geq 0)$$

Exercise

Prove that the sum of k indep. exponential r.v.s (param. λ) is Erlang of order k.

Hint

use induction on k. When $k=1$, Erlang is exponential. On Induction step use convolution to show

$$Y_k = Y_{k-1} + T_k$$

is Erlang of order k.

Compare Bernoulli / Poisson:

	Bernoulli	Poisson
time :	$t \in \{1,2,3,\ldots\}$	$t \in [0,\infty)$
rate :	p/trial	λ/unit time
N_t	Binomial(t,p)	Poisson(λt)
T_k	Geometric(p)	Exponential(λ)
Y_k	Pascal(p, k) ← k = order	Erlang(λ, k) ← k = order