CSE 107 11-25-25

4.5 Sum of a random number of random variables

Let $X_1, X_2, \dots$ be an (infinite) seq. of independent, identically distributed r.v.s (i.i.d.), i.e. They all have same PDF or PMF.

Let X be a r.v. with common dist.

$$P_X = P_{X_i} \quad \text{or} \quad f_X = f_{X_i} \quad \begin{pmatrix} \text{all} \\ i=1,2,\dots \end{pmatrix}$$

Let N be a r.v. with

$$\text{range}(N) \subseteq \{\cancel{0}, 1, 2, \ldots\}$$

<u>Assume $\{N, X_1, X_2, \ldots\}$ is independent</u>. Now define

$$Y = X_1 + X_2 + \cdots + X_N$$

Let $n \geq 1$ be fixed. Then

$$E[Y \mid N = n]$$

$$= E[X_1 + \cdots + X_N \mid N = n]$$

$$= E[X_1 + \cdots + X_n \mid N = n]$$

$$= E[X_1 + \cdots + X_n]$$

$$= E[X_1] + \cdots + E[X_n]$$

$$= n\,E[X]$$

Thus

$$E[Y \mid N] = N\,E[X]$$

By the law of iterated expectation

$$E[Y] = E\big[E[Y|N]\big]$$

$$= E\big[N\,E[X]\big]$$

$$= E[N]\cdot E[X]$$

Also

$$Var(Y|N=n)$$

$$= Var(X_1 + \cdots + X_N \mid N=n)$$

$$= Var(X_1 + \cdots + X_n \mid N=n)$$

$$= \mathrm{Var}(X_1 + \cdots + X_n) \quad \text{by indep.}$$

$$= \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) \quad \text{by indep.}$$

$$= n\,\mathrm{Var}(X)$$

replace n by N:

$$\mathrm{Var}(Y \mid N) = N \cdot \mathrm{Var}(X)$$

By the law of total variance:

$$\operatorname{Var}(Y) = E[\operatorname{Var}(Y|N)] + \operatorname{Var}(E[Y|N])$$

$$= E[N \cdot \operatorname{Var}(X)] + \operatorname{Var}(N \cdot E[X])$$

$$= E[N] \cdot \operatorname{Var}(X) + E[X]^2 \cdot \operatorname{Var}(N)$$

Ex.

We have 2 coins

Coin 1 : $P(\text{head}) = p$

Coin 2 : $P(\text{head}) = q$

flip coin 1 until 1st head.
let N be the # of flips.

note:

$$N \sim \text{Geom}(p)$$

i.e.

$$P_N(n) = \begin{cases} (1-p)^{n-1} \cdot p & \text{if } n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Now flip coin 2 N times and count the # heads. let Y be the # of heads in 2^{nd} sequence. note

$$Y \sim \text{Binomial}(q, N).$$

also

$$Y = X_1 + \cdots + X_N$$

where each X_i is Bernoulli with param. q.

$$X_i \sim \text{Bernoulli}(q)$$

All coin flips are indep. so $\{N, X_1, X_2, \ldots\}$ is indep. let $E[X] = E[X_i]$ and $Var(X) = Var(X_i)$ for all $i = 1, 2, \ldots$. We have

$$E[N] = \frac{1}{p}, \quad Var(N) = \frac{1-p}{p^2}$$

and

$$E[X] = q, \quad Var(X) = q(1-q)$$

Therefore

$$E[Y] = E[N] \cdot E[X] = \frac{q}{p}$$

$$Var(Y) = E[N]\,Var(X) + E[X]^2\,Var(N)$$

$$= \frac{q(1-q)}{p} + \frac{q^2(1-p)}{p^2}$$

$$= \frac{q}{p}\left((1-q) + \frac{q}{p}(1-p)\right)$$

6.1 The Bernoulli Process

Defn

A Bernoulli Process is seq.

$$X_1, X_2, \ldots$$

of i.i.d. Bernoulli r.v.s. i.e.

for all $t \in \{1, 2, 3, \ldots\}$

$$P(X_t = 1) = p \quad \{ \text{head, success, arrival} \}$$

$$P(X_t = 0) = 1 - p \quad \{$$

Can think of Process in 2 ways

- one experiment, repeated ∞-many times

$$\Omega = \{0, 1\}$$

- a single experiment with

$$\Omega = \{\text{infinite seq. of } 0\text{'s}, 1\text{'s}\}$$

$P(\text{any seq.}) = 0$

for instance, let $0 < p < 1$

$$\omega = 111 \cdots\cdots$$

$$P(\omega) = P(\forall t : X_t = 1)$$

$$\leq P(X_t = 1 \text{ for } 1 \leq t \leq n)$$

$$= p^n$$

$\therefore \quad 0 \leq P(\omega) \leq p^n \longrightarrow 0 \quad \text{as } n \to \infty$

$\therefore \quad P(\omega) = 0$

Memorylessness

Let T be the time (# of trials) until 1^{st} success.

Then

$$T \sim Geom(p)$$

so

$$p_T(t) = (1-p)^{t-1} \cdot p \quad (t = 1, 2, \ldots)$$

$$E[T] = \frac{1}{p}$$

$$Var(T) = \frac{1-p}{p^2}$$

Suppose we observe

$$X_1, X_2, \dots, X_n$$

are all failures, for some $n \geq 1$.

Then the event $\{T > n\}$ has occured.

What is the distribution of $T-n$, the time left until the 1st head.

$$p_{T-n|T>n}(t) = P(T-n=t \mid T>n)$$

$$= \frac{P(T-n=t,\ T>n)}{P(T>n)}$$

$$= \frac{P(T=n+t)}{P(T>n)}$$ since $\{T=n+t\} \subseteq \{T>n\}$

$$= \frac{(1-p)^{n+t-1} \cdot p}{(1-p)^{n}}$$

$$= (1-p)^{t-1} \cdot p$$

$$= P(T=t) = p_T(t)$$

Interarrival times

let

T_1 = time to 1^{st} 'arrival'

T_2 = time from T_1 to 2^{nd} arrival

T_3 = " " T_2 " 3^{rd} "

⋮

T_k = " " T_{k-1} to k^{th} arrival

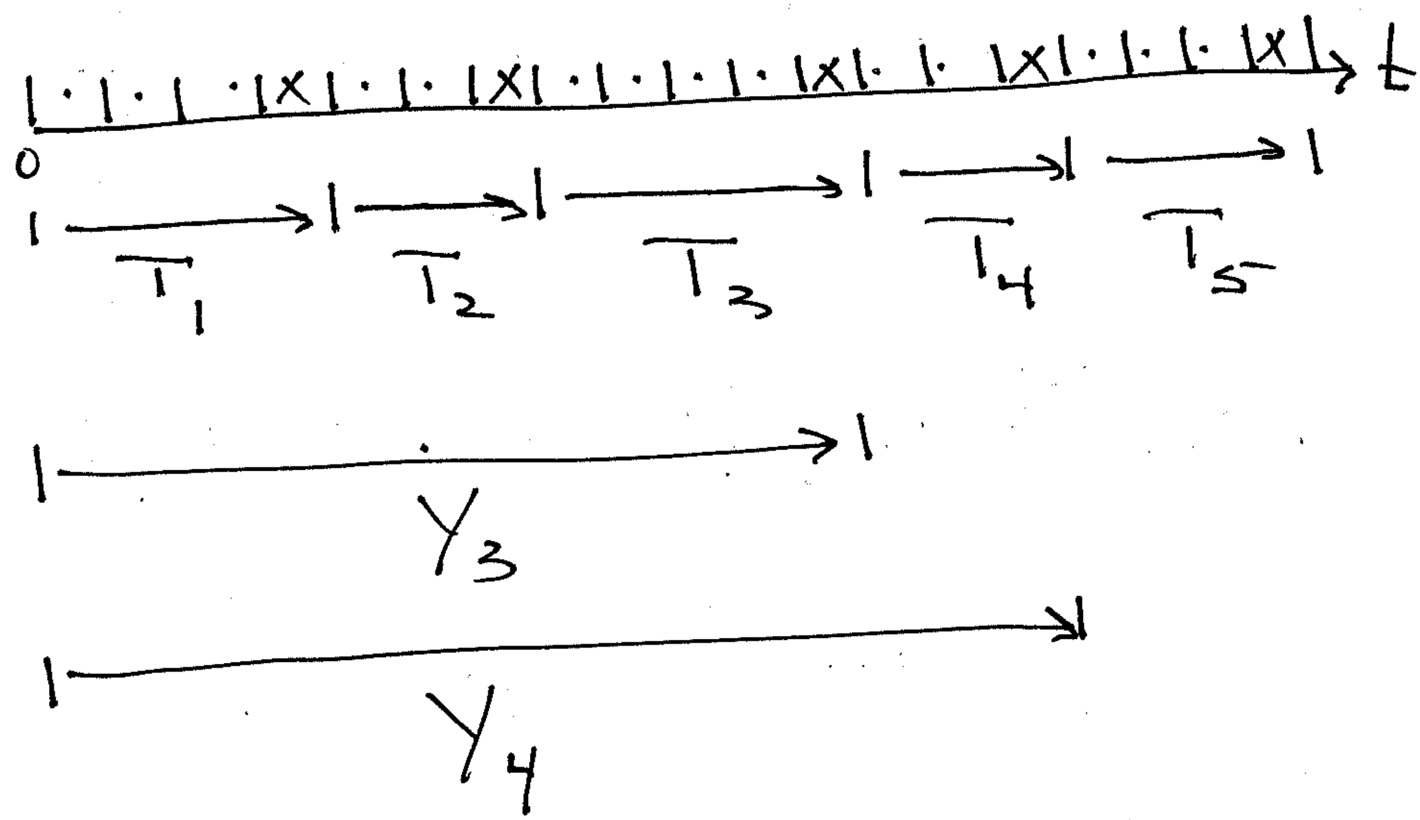
Let $Y_k = T_1 + T_2 + \cdots + T_k$

= time to k^{th} arrival

Alternately

$$T_1 = Y_1$$

$$T_k = Y_k - Y_{k-1} \quad (k = 2, 3, \ldots)$$



By 'fresh start' Property, each

$\overline{T}_K \sim Geom(p)$

what is dist. of Y_K

<u>Exercise</u>

use $Y_K = Y_{K-1} + \overline{T}_K$, convolution, and induction to find P_{Y_K}.

$$Y_1 = \overline{T}_1$$

$$Y_2 = Y_1 + \overline{T}_2$$

$$Y_3 = Y_2 + \overline{T}_3$$

⋮

$$Y_K = Y_{K-1} + T_K$$

answer: Pascal dist.

easy way: next time.