CSE 107 11-20-25

4.3 Conditional Expectation and Variance

Recall

$$g(y) = E[X \mid Y=y]$$

is a fcn of y. substitute Y for y.

$$g(Y) = E[X \mid Y]$$

is a r.v.

Then

$$E[E[X|Y]] = E[g(Y)]$$

$$= \int_{-\infty}^{\infty} g(y) f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} E[X|Y=y] \cdot f_Y(y)\,dy$$

$$= E[X]$$ by the <u>total expectation</u> thm cont. version

Law of Iterated Expectations

$$E[E[X|Y]] = E[X]$$

Similarly, conditional variance:

$$h(y) = Var(X \mid Y=y)$$

$$= E\left[\left(X - E[X \mid Y=y]\right)^2 \mid Y=y\right]$$

substitute Y for y:

$$h(Y) = Var(X \mid Y)$$

conditional variance

$$= E\left[\left(X - E[X \mid Y]\right)^2 \mid Y\right]$$

Law of total Variance

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y])$$

Proof

Recall variance in terms of moments

$$\mathrm{Var}(X|Y) = E[X^2|Y] - E[X|Y]^2$$

Thus

$$E[X^2|Y] = \mathrm{Var}(X|Y) + E[X|Y]^2$$

by law of iterated expectations

$$Var(X) = E[X^2] - E[X]^2$$

$$= E[E[X^2|Y]] - E[E[X|Y]]^2$$

$$= E[Var(X|Y) + E[X|Y]^2]$$

$$- E[E[X|Y]]^2$$

$$= E[Var(X|Y)]$$

$$+ E[E[X|Y]^2] - E[E[X|Y]]^2$$

$$= E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y])$$

$\blacksquare$