CSE 107 11-14-25

Supplemental Lecture

Theorem

if $g:\mathbb{R}\to\mathbb{R}$ is strictly monotonic and differentiable, $h=g^{-1}$ and

$$Y=g(X)$$

then

$$\boxed{f_Y(y)=f_X(h(y))\cdot|h'(y)|}$$

Proof

Suppose g is strictly increasing, so h is also strictly increasing. Then

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(g(X) \le y) \\
&= P(X \le h(y)) \qquad \left\{\begin{array}{l}\text{using that} \\ h \text{ is increasing}\end{array}\right. \\
&= F_X(h(y))
\end{aligned}
$$

Thus upon differentiating

$$
\begin{aligned}
f_Y(y) &= f_X(h(y)) \cdot h'(y) \\
&= f_X(h(y)) \cdot |h'(y)| \qquad \left\{\begin{array}{l}\text{since} \\ h' > 0\end{array}\right.
\end{aligned}
$$

<u>exercise</u>: Prove case when g is is strictly decreasing.

<u>note</u> all definitions and results can be restricted to an interval $I \subseteq \mathbb{R}$.

<u>Ex</u>

let $Y = X^2$ where X is a continuous r.v. with $\text{range}(X) \subseteq [0,\infty)$. Also have $\text{range}(Y) \subseteq [0,\infty)$. Note $g(x) = x^2$ is strictly increasing on $[0,\infty)$, with inverse $h(y) = \sqrt{y}$. Thus

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} \qquad (y > 0)$$

Two random variables $Z = g(X, Y)$

Ex.

let X, Y be indep. cont. uniform on $[0, 1]$. Thus

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

so

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

and similarly for Y. Define Z by

$$Z = \max(X, Y)$$

For $0 \le z \le 1$ we have

$$
\begin{aligned}
F_Z(z) &= P(Z \le z) \\
&= P(\max(X, Y) \le z) \\
&= P(X \le z, Y \le z) \\
&= P(X \le z) \cdot P(Y \le z) \quad \left\{ \begin{array}{l} \text{by indep.} \\ \text{of } X, Y \end{array} \right. \\
&= F_X(z) \cdot F_Y(z) \\
&= z^2
\end{aligned}
$$

hence

$$
f_Z(z) = \begin{cases} 2z & \text{if } 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

Exercise

Let X be cont. unif. on $[a,b]$ and Y cont. unif on $[c,d]$. Find PDF of

$$Z = \max(X, Y)$$

assuming X, Y independent.

Note

- when $b < c$ ($\underset{a}{[}\ \underset{b}{]}\ \underset{c}{[}\ \underset{d}{]}$) then $[a,b] \cap [c,d] = \phi$ and $\max(X,Y) = Y$.
- if $d < a$ ($\underset{c}{[}\ \underset{d}{]}\ \underset{a}{[}\ \underset{b}{]}$), again $[a,b] \cap [c,d] = \phi$ and $\max(X,Y) = X$

so assume both $b \leq c$ and $a \leq d$, and therefore $[a,b] \cap [c,d] \neq \phi$.

Sums $Z = X + Y$ and the Convolution Product

Let X, Y be independent discrete r.v.s and let

$$Z = X + Y$$

Then

$$p_Z(z) = P(Z = z)$$

$$= P(X + Y = z)$$

$$= \sum_{\{(x,y) \mid x+y=z\}} P(X = x, Y = y)$$

$$= \sum_{\{(x, z-x) \mid x \in \text{range}(X)\}} P(X = x, Y = z - x)$$

$$= \sum_x P(X=x)\cdot P(Y=z-x) \quad \left\{ \begin{array}{l} \text{by indep.} \\ \text{of } X, Y \end{array} \right.$$

$$= \sum_x p_X(x)\cdot p_Y(z-x)$$

We call the R.H.S. the <u>convolution product</u> : notation : $(p_X * p_Y)(z)$

Thus

$$\boxed{p_{X+Y} = p_X * p_Y = p_Y * p_X}$$

Now let X, Y independent jointly Cont. r.v.s and

$$Z = X + Y$$

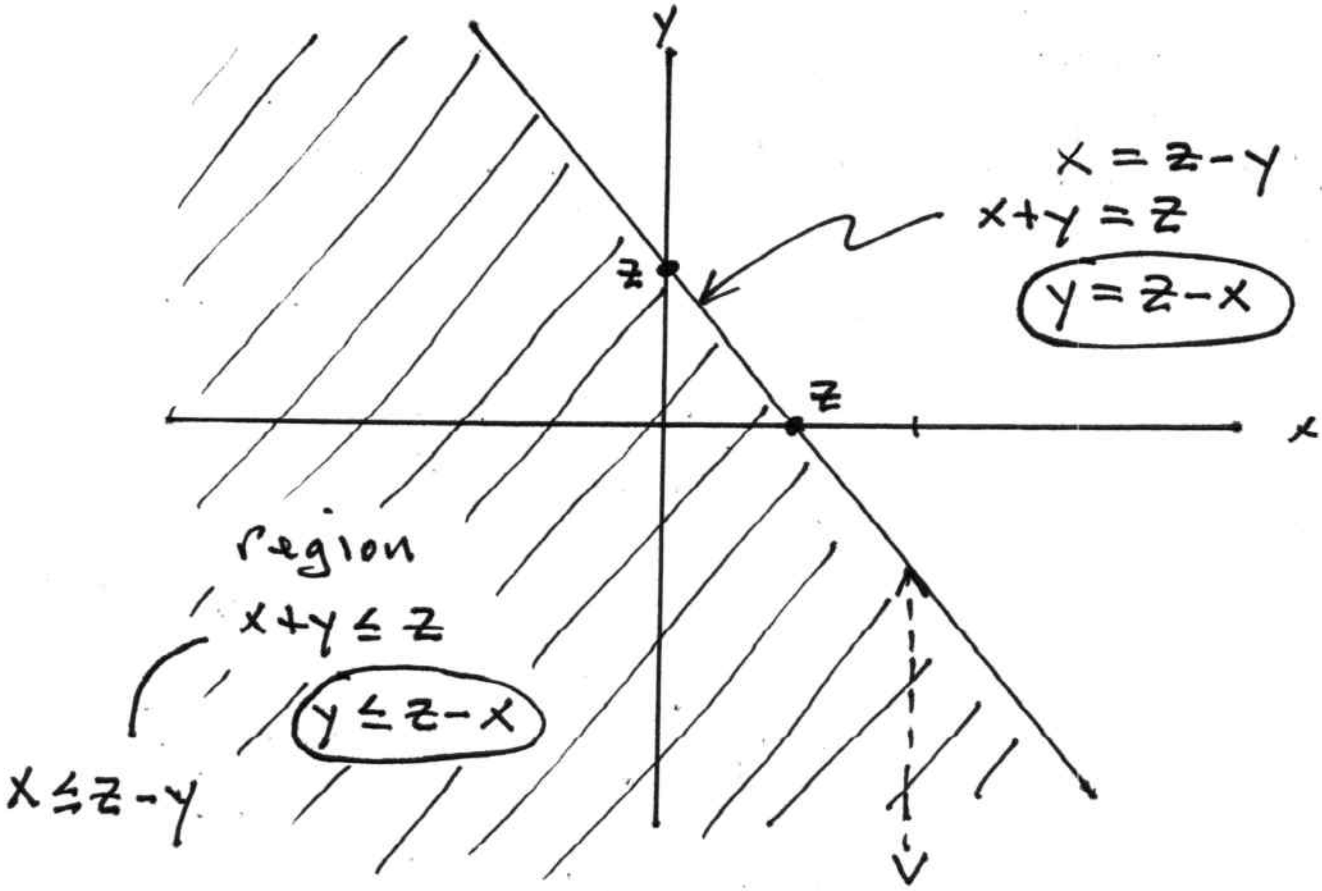
Then

$$F_Z(z) = P(Z \leq z)$$

$$= P(X + Y \leq z)$$

$$= \iint_{\{(x,y) \mid x+y \leq z\}} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{-\infty} \int_{-\infty}^{z-x} f_X(x) f_Y(y)\, dy\, dx \quad \left\{ \begin{array}{l} \text{by} \\ \text{indep.} \end{array} \right.$$



$$= \int_{-\infty}^{\infty} f_X(x) \underbrace{\left(\int_{-\infty}^{z-x} f_Y(y)\, dy \right)}_{P(Y \le z - x)} dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot F_Y(z-x) \cdot dx$$

differentiating w.r.t. z gives

$$f_Z(z) = \frac{d}{dz} \int_{-\infty}^{\infty} f_X(x) \cdot F_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot \frac{d}{dz} F_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx$$

This operation is called the <u>Convolution Product</u> of f_X, f_Y.

<u>notation</u>: $(f_X * f_Y)(z)$

We've shown

$$f_{X+Y}(z) = \left(f_X * f_Y\right)(z)$$

$$= \left(f_Y * f_X\right)(z)$$

Remark

Remember, both discrete and continuous formulas for f_{X+Y} (or P_{X+Y}) <u>require</u> that X and Y be independent.

Ex.

Let X, Y be indep. cont. Unif. on $[a,b]$, i.e.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

and similarly for Y. Let

$$Z = X + Y.$$

Then

$$f_Z(z) = (f_X * f_Y)(z)$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx$$

The integrand is equal to $\left(\frac{1}{b-a}\right)^2$ iff both

$$a \le x \le b \quad \text{and} \quad a \le z-x \le b$$

i.e.

$$a \le x \le b \quad \text{and} \quad z-b \le x \le z-a$$

i.e. iff

$$\max(a, z-b) \le x \le \min(b, z-a)$$

Thus

$$f_Z(z) = \int_{\max(a,z-b)}^{\min(b,z-a)} \frac{1}{(b-a)^2}\, dx$$

$$= \frac{\min(b, z-a) - \max(a, z-b)}{(b-a)^2}$$

The interval of integration will be empty iff either

$$z - a < a \rightarrow z < 2a$$

or

$$z - b > b \rightarrow z > 2b$$

Thus interval is non-empty iff $2a \le z \le 2b$ and

$$f_Z(z) = \begin{cases} \dfrac{\min(b, z-a) - \max(a, z-b)}{(b-a)^2} & \text{if } 2a \le z \le 2b \\ 0 & \text{otherwise} \end{cases}$$

Ex.

Let X, Y be indep. normal r.v.s with

means: μ_X, μ_Y, resp.

Variances: σ_X^2, σ_Y^2, resp.

> 0

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} \cdot e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}}$$

$$f_Y(y) = \ldots \quad \ldots \quad \ldots$$

Let $Z = X + Y$, then

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\cdot\sigma_x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_y} \cdot e^{-\frac{(z-x-\mu_y)^2}{2\sigma_y^2}} \, dx$$

$\vdots$ exercise

$$= \frac{1}{\sqrt{2\pi}\,\sqrt{\sigma_x^2+\sigma_y^2}} \cdot e^{-\frac{(z-\mu_x+\mu_y))^2}{2(\sigma_x^2+\sigma_y^2)}}$$

Thus Z is normal with

mean : $\mu_x + \mu_y$

Variance : $\sigma_x^2 + \sigma_y^2$

It follows that if X, Y are indep. normal, then

$$aX + bY$$

is also normal.