CSE 107

11-13-24

continue example...

e.) Find $P(\overbrace{X \le 2}^{1 \le X \le 2} \mid Y = 3)$

$$P(X \le 2 \mid Y = 3) = \int_1^2 f_{X|Y}(x|3)\,dx$$

$$= \frac{\int_1^2 \frac{1}{\sqrt{2\pi}} e^{-(x-3)^2/2}\,dx}{\Phi(0) - \Phi(-2)}$$

$$\begin{cases} u = x-3 \\ du = dx \\ x=1 \rightarrow u=-2 \\ x=2 \rightarrow u=-1 \end{cases}$$

$$= \frac{\int_{-2}^{-1} \frac{1}{\sqrt{2\pi}} \cdot e^{-u^2/2}\,du}{\Phi(0) - \Phi(-2)}$$

$$= \frac{\Phi(-1) - \Phi(-2)}{\Phi(0) - \Phi(-2)}$$

$$= \boxed{.2848}$$

d.) Find $E[Y]$.

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} y \cdot \left(\frac{1}{2} \int_{1}^{3} \frac{e^{-(y-x)^2/2}}{\sqrt{2\pi}}\, dx \right) dy$$

$$= \frac{1}{2} \int_{1}^{3} \underbrace{\left(\int_{-\infty}^{\infty} y \cdot \frac{e^{-(y-x)^2/2}}{\sqrt{2\pi}} \, dy \right)}_{x} dx$$

$$= \frac{1}{2} \int_{1}^{3} x \, dx = \frac{1}{2} \frac{1}{2} x^2 \Big|_{1}^{3}$$

$$= \frac{1}{4}(9-1) = \boxed{2}$$

4.1 Derived Distributions

let $g: \mathbb{R} \to \mathbb{R}$, $Y = g(X)$,

X, Y cont. or disc. r.v.s.

Goal: compute $f_Y(y)$ from $f_X(x)$. We call $f_Y(y)$ a derived distribution.

2-step Process

1. find CDF $F_Y(y)$:

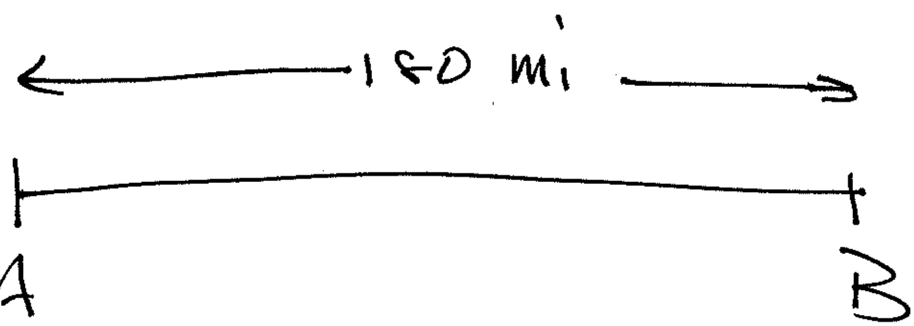
$$F_Y(y) = P(Y \leq y)$$
$$= P(g(X) \leq y)$$
$$= \int_{\{x \mid g(x) \leq y\}} f_X(x)\,dx$$

2. differentiate

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Ex.

A car travels a dist. of 180 miles at const. rate R, where R is a r.v. uniform cont. on $[30, 60]$. (mph).



Let T be time of travel.

Find $f_T(t)$.

Soln

From $180 = R \cdot T$ we have

$$T = \frac{180}{R}$$

① $F_T(t) = P(T \le t)$

$$= P\left(\frac{180}{R} \le t\right)$$

$$= P\left(R \ge \frac{180}{t}\right)$$

$$= 1 - P\left(R < \frac{180}{t}\right)$$

$$= 1 - F_R\left(\frac{180}{t}\right)$$

② differentiate.

$$f_T(t) = -f_R\left(\frac{180}{t}\right)\cdot\left(-\frac{180}{t^2}\right)$$

Recall $f_R(r) = \begin{cases} 1/30 & 30 \le r \le 60 \\ 0 & \text{otherwise} \end{cases}$

$$\therefore\ f_T(t) = \begin{cases} \frac{1}{30}\cdot\frac{180}{t^2} & \text{if } 30 \le \frac{180}{t} \le 60 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{6}{t^2} & \frac{1}{60} \le \frac{t}{180} \le \frac{1}{30} \\ 0 & \text{---} \end{cases}$$

$$\boxed{f_T(t) = \begin{cases} \frac{6}{t^2} & \text{if } 3 \le t \le 6 \\ 0 & \text{otherwise} \end{cases}}$$

If $Y = aX + b$, then
$(a \neq 0)$

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

Proof

(1) $F_Y(y) = P(Y \leq y)$

$$= P(aX + b \leq y)$$

$$= \begin{cases} P\left(X \leq \frac{y-b}{a}\right) & (a>0) \\ P\left(X \geq \frac{y-b}{a}\right) & a<0 \end{cases}$$

$$= \begin{cases} P\left(X \leq \frac{y-b}{a}\right) & (a>0) \\ 1 - P\left(X < \frac{y-b}{a}\right) & (a<0) \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right) & (a>0) \\ 1-F_X\left(\frac{y-b}{a}\right) & (a<0) \end{cases}$$

② $$f_Y(y) = \begin{cases} \frac{1}{a} f_X\left(\frac{y-b}{a}\right) & (a>0) \\ -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) & (a<0) \end{cases}$$

$$\therefore \boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \quad (a \neq 0)}$$

Ex

Let X be normal(μ, σ^2), then $Y = aX + b$ is normal$(a\mu + b, a^2\sigma^2)$.

Proof

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(x-\mu)^2/2\sigma^2}.$$

Thus

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,|a|\sigma} \cdot e^{-\left(\frac{y-b}{a} - \mu\right)^2/2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi}\,|a\sigma|} \cdot e^{-\left(\frac{y-b-a\mu}{a}\right)^2 / 2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi}\,|a\sigma|} \cdot e^{-(y-(a\mu+b))^2 / 2(a\sigma)^2}$$

Thus Y is normal with mean $a\mu+b$ and variance $(a\sigma)^2$ (i.e. std. dev. $|a|\sigma$.)

$\blacksquare$

<u>Defn</u>

$g : \mathbb{R} \to \mathbb{R}$ is called <u>strictly monotonic</u> iff for all $x_1, x_2 \in \mathbb{R}$ either:

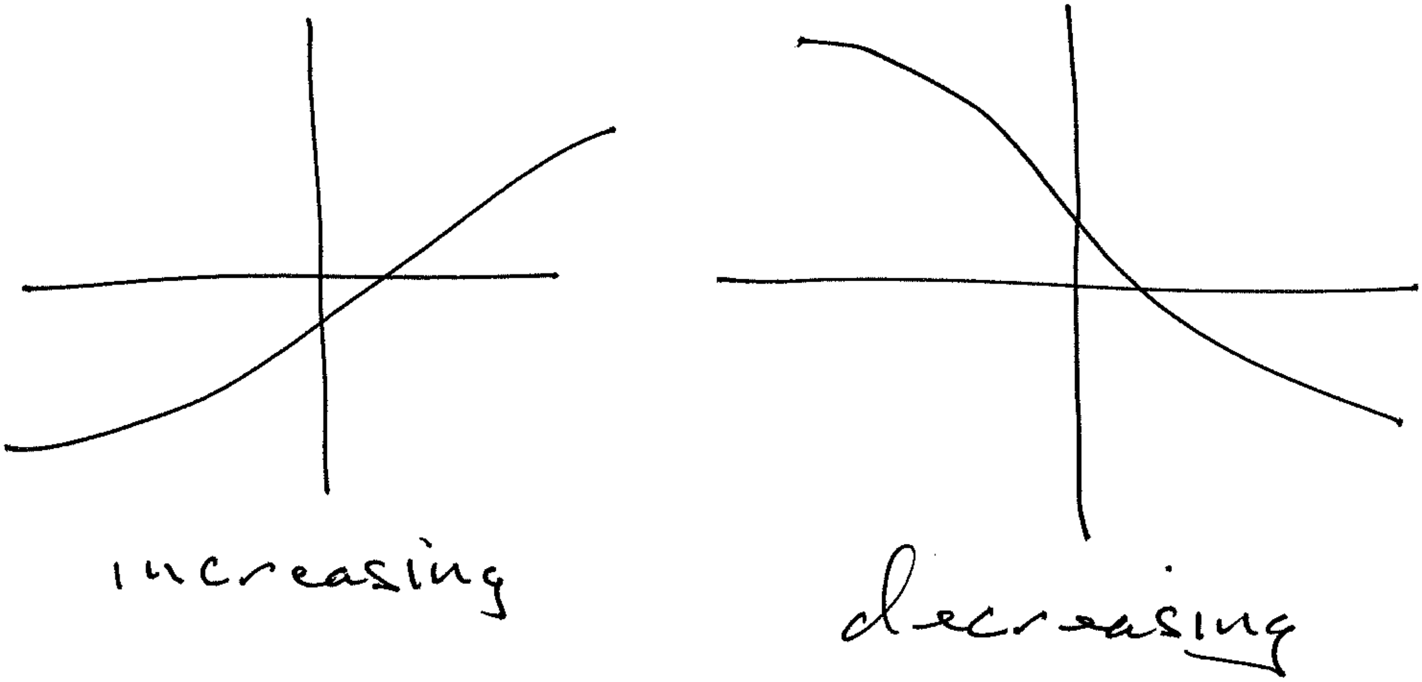
(1) $x_1 < x_2 \Rightarrow g(x_1) < g(x_2)$ (increasing)

or

(2) $x_1 < x_2 \Rightarrow g(x_1) > g(x_2)$ (decreasing)

<u>note</u> we can restrict definition to an interval $[a,b]$ so

$$g : [a,b] \to \mathbb{R}$$



<u>note</u>: such functions are invertible!

Let $h = g^{-1}$, i.e.

$$h(g(x)) = x \quad \text{for all } x \in \mathbb{R}$$

and

$$g(h(y)) = y \quad \text{"} \quad \text{"} \quad y \in \mathbb{R}$$

We can show (exercise)

(1) g strict. inc. $\Rightarrow h = g^{-1}$ strict inc.

and

(2) g strict dec. $\Rightarrow h = g^{-1}$ strict dec.

If g is differentiable, then so is $h = g^{-1}$ and

(1) $\Leftrightarrow g' > 0 \Leftrightarrow h' > 0$

and

(2) $\Leftrightarrow g' < 0 \Leftrightarrow h' < 0$

Now let g be strict monotone & differentiable. Then if

$$Y = g(X)$$

then

$$\boxed{f_Y(y) = f_X(h(y)) \cdot |h'(y)|}$$