CSE 107 11-10-25

Supplemental Lecture

Independence

Let X, Y be jointly continuous r.v.s

Defn

X, Y are independent iff

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

for all $x, y \in \mathbb{R}$.

Equivalently

- $f_{X|Y}(x|y) = f_X(x)$ $\begin{cases} \text{for all } y \text{ s.t. } f_Y(y) > 0 \\ \text{and all } x \end{cases}$

- $f_{Y|X}(y|x) = f_Y(y)$ $\begin{cases} \text{for all } x \text{ s.t. } f_X(x) > 0 \\ \text{and all } y \end{cases}$

Generalization to 3 or more r.v.s

$$f_{X,Y,Z}(x,y,z) = f_X(x) \cdot f_Y(y) \cdot f_Z(z)$$

If X, Y are independent, then any any events definable in terms of X and Y are also independent:

$$\{X \in I\} \text{ and } \{Y \in J\}$$

In particular, if $I = (-\infty, x)$, $J = (-\infty, y)$:

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds\, dt$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_X(t) f_Y(s)\, ds\, dt$$

$$= \int_{-\infty}^{x} f_X(t) \left(\int_{-\infty}^{y} f_Y(s)\, ds \right) dt$$

$$= \int_{-\infty}^{x} f_X(t)\, dt \cdot \int_{-\infty}^{y} f_Y(s)\, ds$$

$$= P(X \le x) \cdot P(Y \le y)$$

$$= F_X(x) \cdot F_X(y)$$

<u>Rmk</u>: Converse is true, i.e.

$$F_{X,Y} = F_X \cdot F_Y \underset{\Longleftarrow}{\Longrightarrow} f_{X,Y} = f_X \cdot f_Y$$

showed

As in the discrete case, we have

(1) $E[XY] = E[X] \cdot E[Y]$

(2) $E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)]$

(3) $Var(X+Y) = Var(X) + Var(Y)$

<u>Exercise</u>

- Prove (1) (mimic notes notes-10/21/25-Page 6.)

- Prove (2) (Hard: Prob. 44 on P.134, adapt to continuous case)

- Prove (3) (Easy!)

3.6 Baye's Rule

Recall

$$P(A|B) = \frac{P(B|A)\cdot P(A)}{P(B)}.$$

If $A_1, \ldots, A_n$ form a Partition of Ω, by total Prob. thm

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

$$= \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

so for any $k \in \{1, 2, \ldots, n\}$

$$\boxed{P(A_k|B) = \frac{P(B|A_k)\cdot P(A_k)}{\sum_{i=1}^{n} P(B|A_i)\cdot P(A_i)}}$$

If X, Y are discrete r.v.s, this becomes

$$\boxed{p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)\cdot p_X(x)}{\sum_{t\,\in\,\mathrm{range}(X)} p_{Y|X}(y|t)\cdot p_X(t)}}$$

In the continuous this becomes

$$\boxed{f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)\cdot f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|t)\cdot f_X(t)\, dt}}$$

Ex.

Let Y be a normal r.v. with $\sigma = 1$ and mean X, another r.v. that is continuous uniform on $[1, 3]$.

a) find $f_Y(y)$

Solution

We are given

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2} & \text{if } \begin{array}{l} 1 \le x \le 3 \\ y \in \mathbb{R} \end{array} \\ 0 & \text{otherwise} \end{cases}$$

so

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_{Y|X}(y|x)\, dx$$

$$= \frac{1}{2} \int_1^3 \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-y)^2}{2}} \, dx \quad \begin{cases} u = x - y \\ du = dx \end{cases}$$

$$= \frac{1}{2} \int_{1-y}^{3-y} \frac{1}{\sqrt{2\pi}} \cdot e^{-u^2/2} \, du$$

$$= \frac{1}{2} \left(\Phi(3-y) - \Phi(1-y) \right) \qquad ///$$

b.) find $f_{X|Y}(x|y)$

solution

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot f_X(x)}{f_Y(y)}$$

$$= \begin{cases} \dfrac{\frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2}}{\Phi(3-y) - \Phi(1-y)} & \text{if } 1 \le x \le 3,\ y \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

c.) Suppose we sample Y and get $y=3$. What is the probability that $X \le 2$, i.e. find

$$P(X \le 2 \mid Y = 3).$$

Exercise: write integral, do ch. of. var. $\begin{cases} u = x-3 \\ du = dx \end{cases}$

d.) Find $E[Y]$

Exercise: do this.