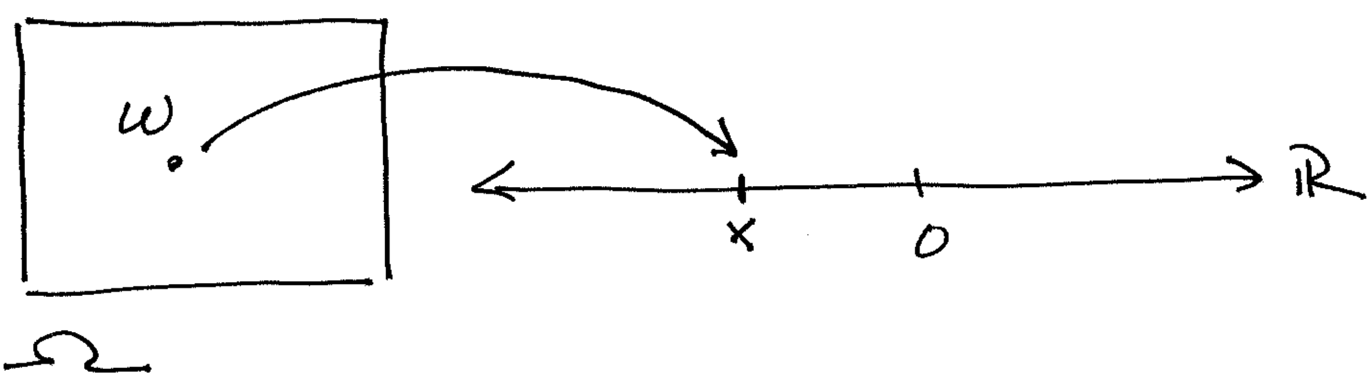
CSE 107 10-9-25

2.1 Random Variables

Defn

A <u>random variable</u> is a function

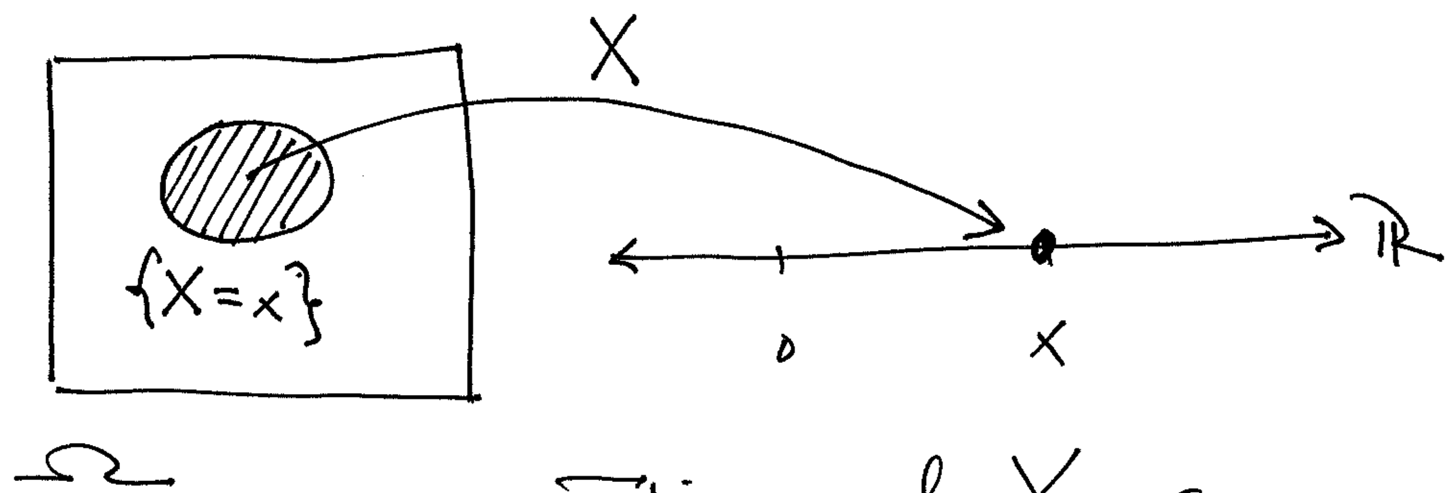
$$X : \Omega \to \mathbb{R}$$



notation: $X(\omega) = x$

- <u>notation</u>:

$$\{X=x\} = \{\omega\in\Omega \mid X(\omega)=x\}$$

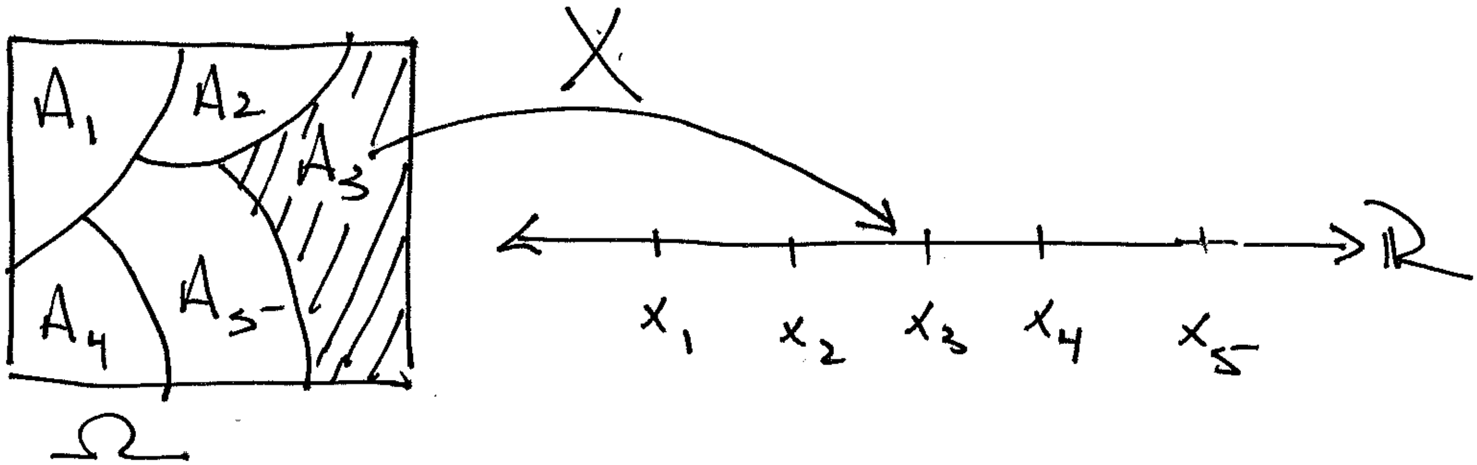


Think of X as transfering 'mass'.

- X is called <u>discrete</u> if its range

$$\text{range}(X) = \{X(\omega) \mid \omega\in\Omega\} \subseteq \mathbb{R}$$

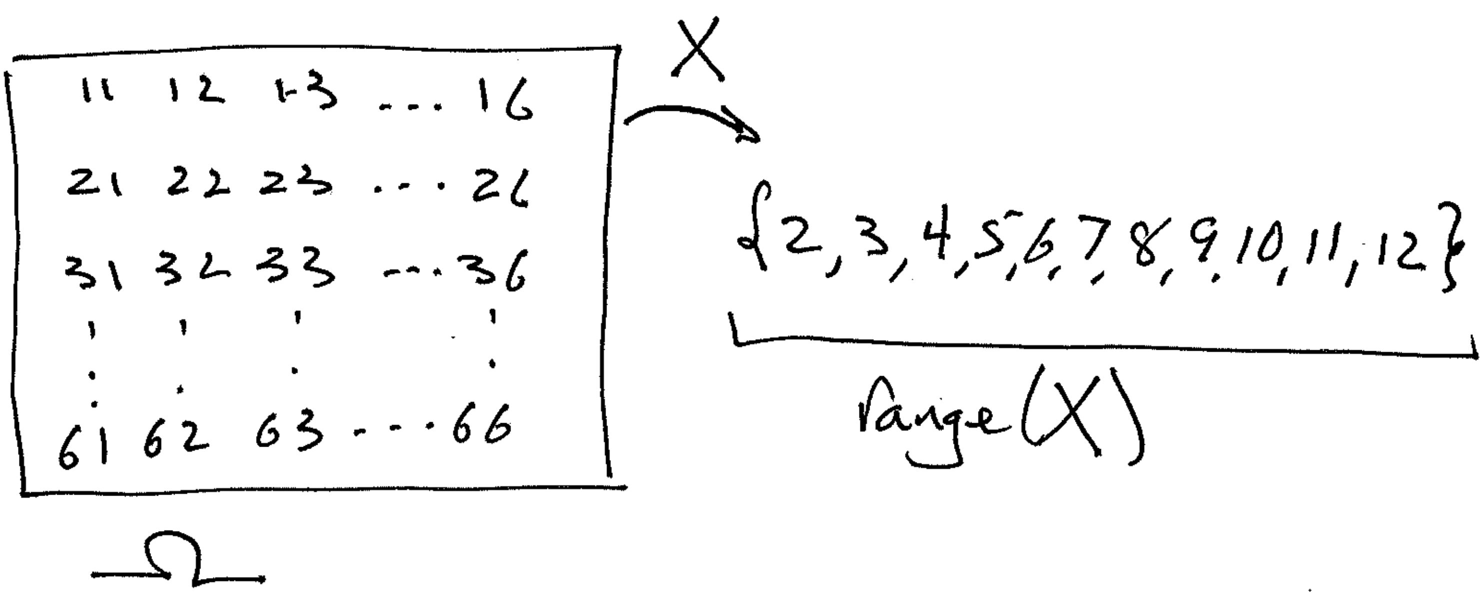
is a discrete subset of $\mathbb{R}$.



$$A_i = \{X = x_i\} \qquad (i = 1, 2, 3, 4, 5)$$

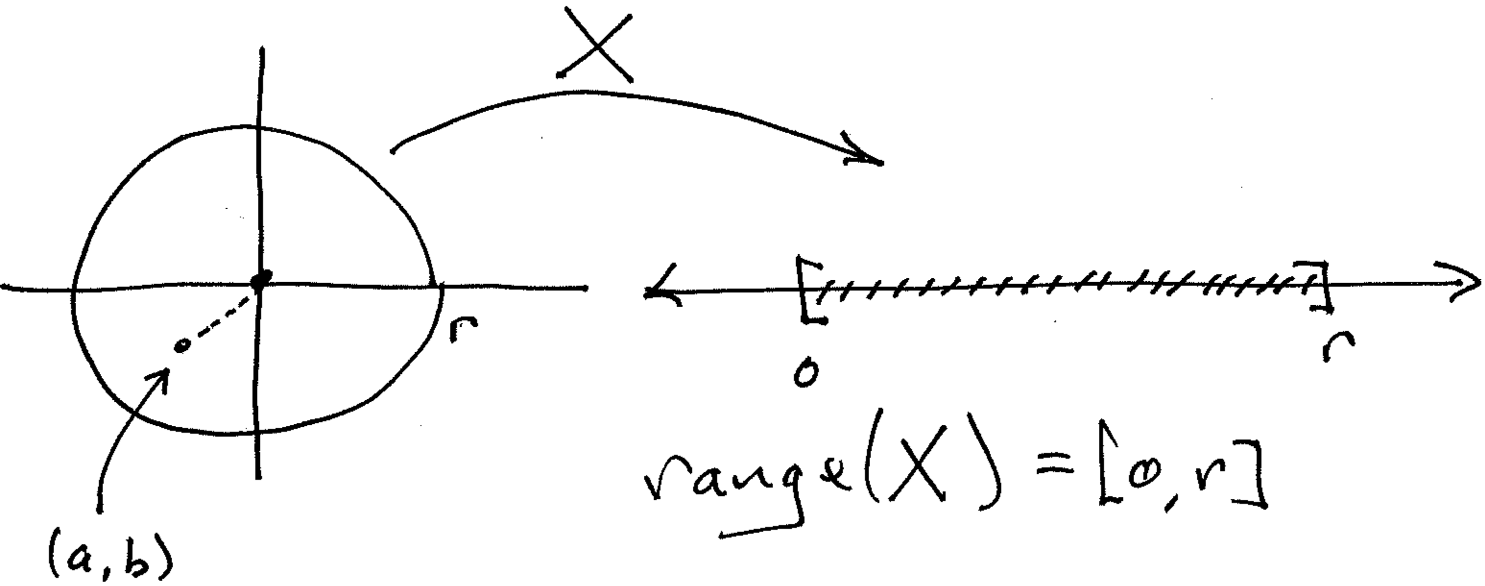
Ex. Roll 2 fair independent dice.

let $X((i,j)) = i + j$.



Ex. throw a dart at circular target centered at (0,0

let $X(a,b) = \sqrt{a^2+b^2}$



$\text{range}(X) = [0,r]$

This is a continuous r.v.

2.2 Probability mass function (PMF)

Defn

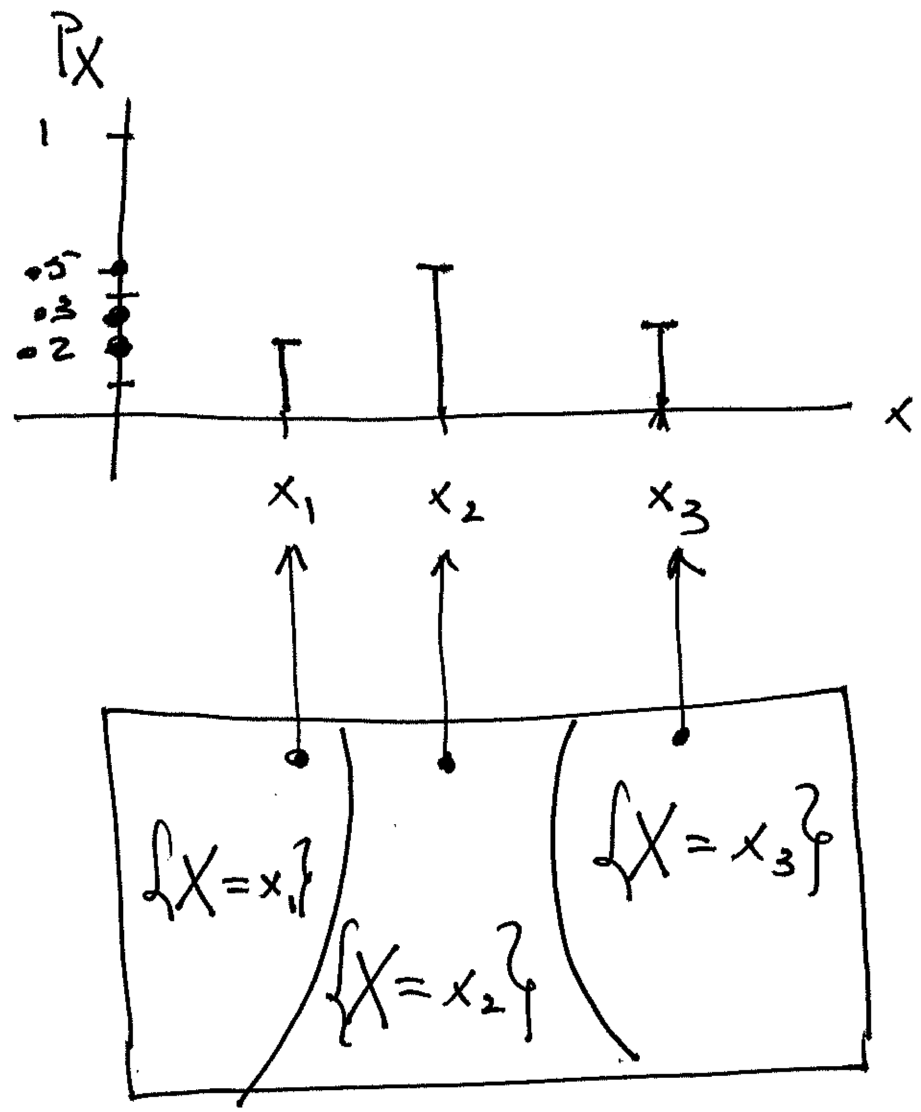
let X be a discret r.v. The Probability Mass Function (PMF) of X is

$$p_X(x) = P(\{X=x\}) = P(X=x)$$

observe $p_X : \mathbb{R} \to [0,1]$. also note

$$\sum_{x \in \text{range}(X)} p_X(x) = 1$$

Ex.



$$P_X(x) = \begin{cases} .2 & \text{if } x = x_1 \\ .5 & \text{if } x = x_2 \\ .3 & \text{if } x = x_3 \\ 0 & \text{otherwise} \end{cases}$$

Bernoulli r.v.

Perform 1 Bernoulli trial. let

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \text{ is success} \\ 0 & \text{if } \omega \text{ is failure} \end{cases}$$

Its PMF is

$$P_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \end{cases}$$

where $p = P(\text{success})$, is called the Parameter of the r.v. X.

Binomial Random Variable

Perform a sequence of independent Bernoulli trials of parameter p.

Let

$$X(\omega) = \#\text{ of successes in } n \text{ trials}$$

we get Binomial Probabilities

$$P_X(k) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, 1, 2, \ldots, n$, and 0 otherwise.

<u>note</u> $\sum_{k=0}^{n} P_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p+(1-p))^n = 1$

✓

The Geometric Random Variable

Perform independent Bernoulli trials (param. p) until 1^{st} success.

Let

$$X(\omega) = \# \text{ of trials until } 1^{st} \text{ success}$$

note: $\Omega = \{1, 01, 001, 0001, \ldots, 00\ldots0\ldots\}$

$$X(\omega) = \text{length}(\omega).$$

$$\text{range}(X) = \{1, 2, 3, \ldots\} \cup \{\infty\}$$

Recall

$$P_X(k) = P(X=k)$$

is the prob. of getting a seq. of $k-1$ failures followed by 1 success. Thus

$$P_X(k) = (1-p)^{k-1} \cdot p$$

for $k = 1, 2, 3, \ldots\ldots$, and 0 otherwise.

<u>note</u>

$$\sum_{k=1}^{\infty} P_X(k) = \sum_{k=1}^{\infty} p \cdot (1-p)^{k-1} = p \sum_{k=1}^{\infty} (1-p)^{k-1}$$

$$= p \cdot \sum_{k=0}^{\infty} (1-p)^{k} = p \left(\frac{1}{1-(1-p)} \right)$$

$$= \cancel{p} \cdot \frac{1}{\cancel{p}} = \boxed{1} \checkmark$$

The Poisson Random Variable

A Poisson r.v. has PMF

$$P_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad (\text{for } k = 0, 1, 2, \ldots)$$

and 0 otherwise. (here $\lambda > 0$ is the parameter)

observe

$$\sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda}$$

$$= e^{-\lambda + \lambda} = e^0 = 1.$$

<u>Note</u>: Poisson is the continuous time limit of Binomial.

We have the approximation

$$e^{-\lambda}\cdot\frac{\lambda^k}{k!}\approx\binom{n}{k}p^k(1-p)^{n-k}\quad(k=0,\dots,n)$$

where $\lambda=np$, n large, p small.

More Precisely let $\lambda>0$ be constant. let $p=\frac{\lambda}{n}$, then

$$\boxed{\lim_{n\to\infty}\binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}=e^{-\lambda}\cdot\frac{\lambda^k}{k!}}$$