CSE 107 10-7-25

<u>Exercise</u>

let $A, B \subseteq \Omega$. Prove that the the following are equivalent.

(a) A, B are independent.

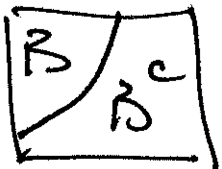
(b) A, B^c " "

(c) A^c, B " "

(d) A^c, B^c " "

Proof of (a) $\Rightarrow$ (b)

Partition B, B^c of Ω



$$\therefore\ A = (A \cap B) \cup (A \cap B^c)$$

$\uparrow$ disjoint union

$$\therefore\ P(A) = P(A \cap B) + P(A \cap B^c) \quad \text{by axiom 3}$$

$$P(A) = P(A) \cdot P(B) + P(A \cap B^c) \quad \text{by (a)}$$

$$\therefore\ P(A \cap B^c) = P(A) - P(A) \cdot P(B)$$

$$= P(A)(1 - P(B))$$

$$= P(A) \cdot P(B^c) \quad \blacksquare$$

Conditional Independence

Recall if $P(C) > 0$, then

$P(\cdot \mid C)$ is a valid Prob. law.

Defn

We say $A, B \subseteq \Omega$ are

conditionally independent given C

iff

$$\boxed{P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)}$$

Warning

$$\boxed{\text{Cond. indep. of } A, B \text{ w.r.t. } C} \quad \begin{matrix} \not\Rightarrow \\ \not\Leftarrow \end{matrix} \quad \boxed{\text{Uncond. indep. of } A, B}$$

See examples 1.20, 1.21 on p. 37

There is an interesting equiv. definition of cond. indep.

note

$$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)}$$

$$= \frac{\cancel{P(C)} \cdot P(B \mid C) \cdot P(A \mid B \cap C)}{\cancel{P(C)}}$$

by the mult. law

$$= P(B \mid C) \cdot P(A \mid B \cap C)$$

Def. of cond. indep. given C:

$$\cancel{P(B \mid C)} \cdot P(A \mid B \cap C) = P(A \mid C) \cdot \cancel{P(B \mid C)}$$

Alternate defn of cond. indep:

$$\boxed{P(A|B\cap C) = P(A|C)}$$

cond. indep of A,B given C.

<u>note</u>

$$P(A|B|C) = \frac{P(A\cap B|C)}{P(B|C)}$$

$$= \frac{P(A\cap B\cap C)/\cancel{P(C)}}{P(B\cap C)/\cancel{P(C)}}$$

$$= P(A|B\cap C)$$

Independence of multiple events

Defn

We say A, B, C are <u>independent</u> iff the following are true.

$$\text{(1) } P(A\cap B\cap C) = P(A)\cdot P(B)\cdot P(C)$$

$$\left[\begin{array}{l} \text{(2) } P(A\cap B) = P(A)\cdot P(B) \\ \text{(3) } P(B\cap C) = P(B)\cdot P(C) \\ \text{(4) } P(A\cap C) = P(A)\cdot P(C) \end{array}\right]$$

<u>Pairwise independent</u>

Warning

$$\boxed{\begin{array}{l}\text{Pairwise} \\ \text{independence} \\ \text{of } A, B, C\end{array}} \;\begin{array}{c}\not\Rightarrow \\ \not\Leftarrow\end{array}\; \boxed{\begin{array}{l}(1)\ P(A \cap B \cap C) \\ \quad = P(A) \cdot P(B) \cdot P(C)\end{array}}$$

See examples 1,22, 1.23 on p. 39 of text.

Defn

In general we say $\{A_1, A_2, \ldots, A_n\}$ of events is <u>independent</u> iff

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdots P(A_{i_k})$$

for every $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$

Binomial Probabilities

Defn

an experiment with 2 possible outcomes is called a **Bernoulli trial**.

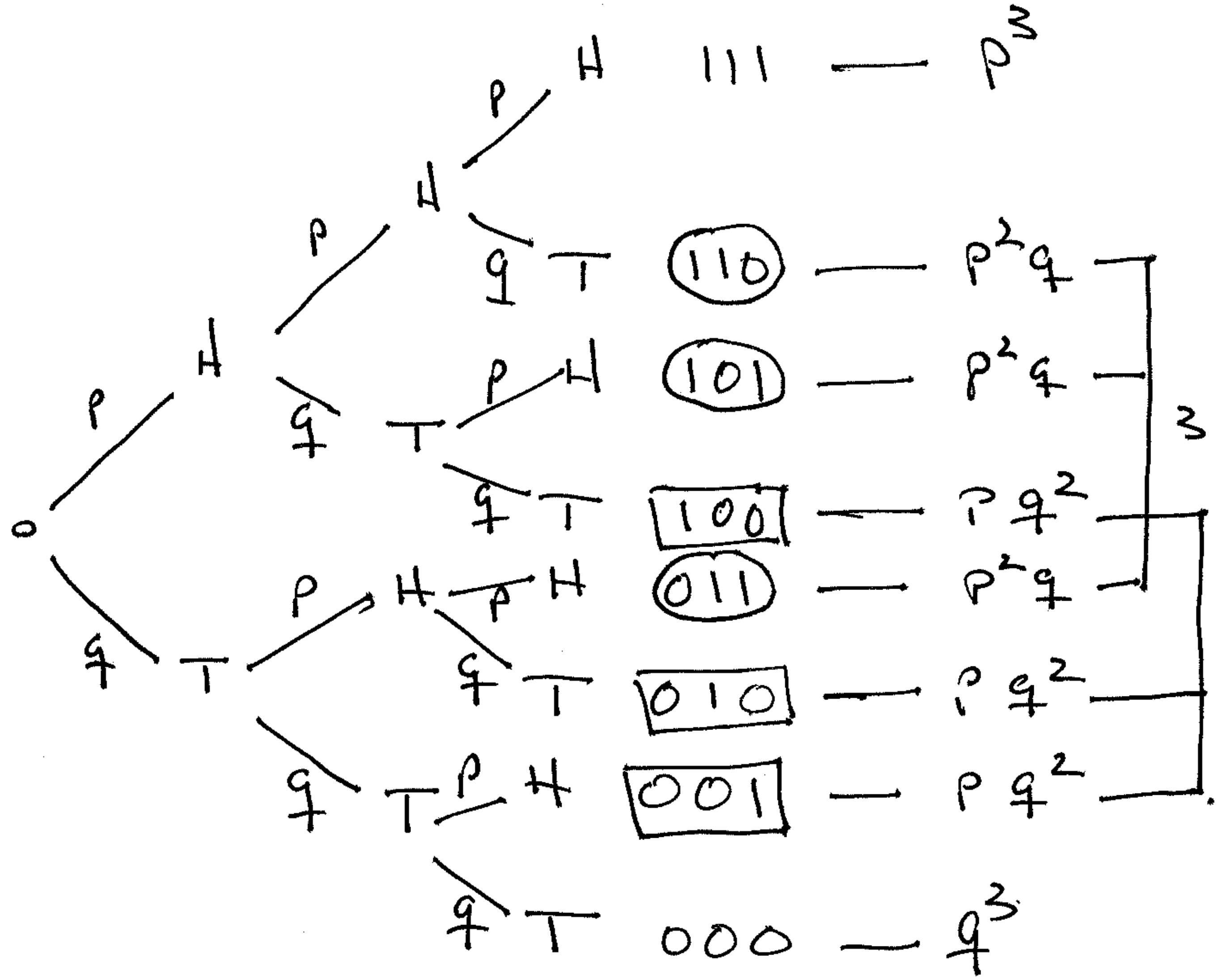
$$\text{outcomes} = \begin{cases} \text{success} = 1 = \text{heads} \\ \text{failure} = 0 = \text{tails} \end{cases}$$

Ex.

· Perform 3 independent tosses of a weighted coin.

$$P(H) = p, \quad P(T) = 1-p = q$$

outcomes Prob



note

$$p^3 + 3p^2q + 3pq^2 + q^3 = (p+q)^3 = 1^3 = 1$$

$$\binom{3}{3} = \frac{3!}{3!\,0!} = 1, \quad \binom{3}{2} = \frac{3!}{2!\,1!} = 3, \quad \binom{3}{1} = \frac{3!}{1!\,2!} = 3$$

$$\binom{3}{0} = \frac{3!}{0!\,3!} = 1$$

$$P(3 \text{ heads}) = p^3$$

$$P(2 \text{ heads}) = 3p^2q$$

$$P(1 \text{ head}) = 3pq^2$$

$$P(0 \text{ heads}) = q^3$$

In general, if we do n independent Bernoulli trials with $P(\text{success}) = p$, then

$$\boxed{P(\#\text{successes} = k) = \binom{n}{k} p^k (1-p)^{n-k}}$$

Binomial Probabilities

for $k = 0, 1, 2, \ldots n$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Binomial coefficient

$$\binom{n}{k} = \left[\begin{array}{l}\text{\# of } k\text{-subsets} \\ \text{of an } n\text{-set}\end{array}\right]$$

$$= \left[\begin{array}{l}\text{\# bit strings of length } n \\ \text{containing } k \text{ 1's.}\end{array}\right]$$

$$= \frac{n!}{k!\,(n-k)!}$$

(1.6) Counting – Combinatorics

Read 1.6, review cse 16.

- # permutations of n elements: $n!$
- # k-permutations of n elements:

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

- # k-combinations of n elements:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

= (# k-subsets of an n-set)

= (# bit strs of len. n, with exactly k 1-s)

Ex. $n = 4, \; k = 2$

$$S = \{1, 2, 3, 4\}$$

bit strings		subsets
0000	—	ϕ
0001	—	$\{4\}$
0010	—	$\{3\}$
$\boxed{0011}$	—	$\boxed{\{3,4\}}$
0100	—	$\{2\}$
$\boxed{0101}$	—	$\boxed{\{2,4\}}$
$\boxed{0110}$		⋮
0111		
1000		
$\boxed{1001}$		
$\boxed{1010}$		
1011		
$\boxed{1100}$		
1101		
1110		
1111		