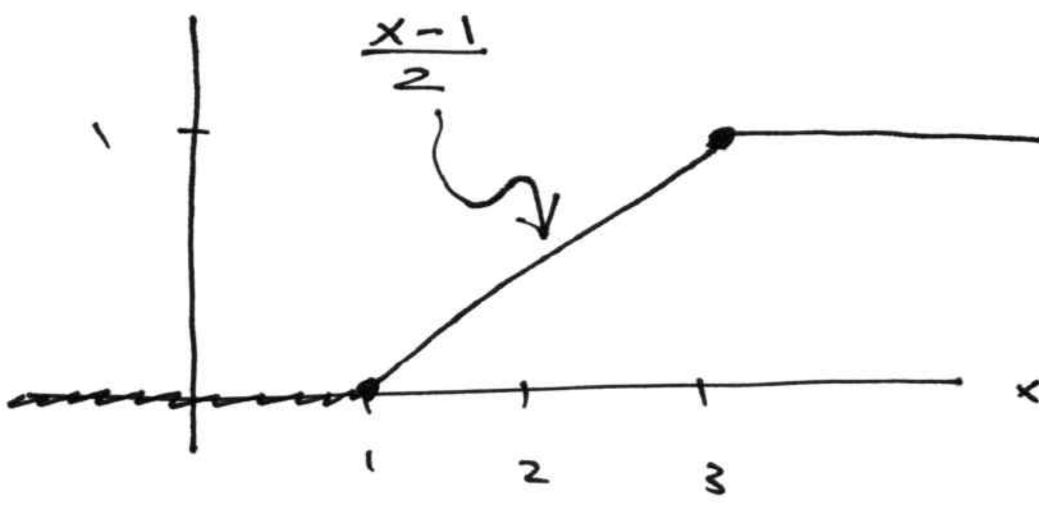
CSE 107 10-31-25

Supplemental Lecture

Ex. (Continue) cont. unif on [1, 3]

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{x-1}{2} & 1 \le x \le 3 \\ 1 & x > 3 \end{cases}$$



- note: the CDF

$$F_X(x) = P(X \leq x)$$

is monotonically non-decreasing, i.e.

$$x_1 \leq x_2 \implies F_X(x_1) \leq F_X(x_2)$$

- if X is Discrete, F_X is Piecewise constant

$$F_X(k) = \sum_{i \leq k} P_X(i)$$

also

$$p_X(k) = P(X = k)$$

$$= P(X \le k) - P(X \le k-1)$$

$$= F_X(k) - F_X(k-1)$$

- if X is continuous, then $F_X(x)$ is a continuous function

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$\therefore \quad f_X(x) = \frac{d}{dx} F_X(x)$$

Ex.

you can take a test 3 times, and your final score X is the maximum of your 3 scores.

$$X = \max(X_1, X_2, X_3)$$

where X_i $(i=1,2,3)$ is score on test i. Assume scores on different tests are independent.

suppose X_i is discrete uniform on $\{1,2,3,4,5,6,7,8,9,10\}$, i.e.

$$P_{X_i}(k) = \begin{cases} \frac{1}{10} & \text{if } k=1,2,3,4,\ldots,10 \\ 0 & \text{otherwise} \end{cases}$$

Find the PMF of X.

Solution

first find CDF $F_X(k)$, then difference to get PMF

$$P_X(k) = F_X(k) - F_X(k-1)$$

observe

$$F_{X_i}(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1/10 & 1 \le x < 2 \\ 2/10 & 2 \le x < 3 \\ 3/10 & 3 \le x < 4 \\ \vdots & \vdots \\ 9/10 & 9 \le x < 10 \\ 1 & x \ge 10 \end{cases}$$

so $F_{X_i}(k) = \frac{k}{10}$ for $k = 1, 2, 3, \ldots, 10$

so

$$F_X(k) = P(X \le k)$$

$$= P(\max(X_1, X_2, X_3) \le k)$$

$$= P(X_1 \le k, X_2 \le k, X_3 \le k)$$

$$= P(X_1 \le k) \cdot P(X_2 \le k) \cdot P(X_3 \le k)$$

by independence

$$= F_{X_1}(k) \cdot F_{X_2}(k) \cdot F_{X_3}(k)$$

$$= \left(\frac{k}{10}\right)^3 \quad (\text{for } k = 1, 2, 3, \ldots, 10)$$

$$\therefore\ P_X(k) = \begin{cases} \left(\frac{k}{10}\right)^3 - \left(\frac{k-1}{10}\right)^3 & \text{if } k = 1, 2, \ldots, 10 \\ 0 & \text{otherwise} \end{cases}$$

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Compare Geometric & Exponential R.V.s

Recall

- if X is Geometric(p)

$$P_X(k) = p(1-p)^{k-1} \quad (k = 1, 2, \ldots)$$

Thus

$$F_X(n) = \sum_{k=1}^{n} p(1-p)^{k-1}$$

$$= p \cdot \sum_{k=1}^{n} (1-p)^{k-1}$$

$$= \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{1-(1-p)}}$$

$$= 1-(1-p)^n$$

- if Y is Exponential(λ) then

$$f_Y(x) = \lambda e^{-\lambda x} \quad (\text{if } x \geq 0)$$

Thus if $x \geq 0$

$$F_Y(x) = \int_0^x \lambda e^{-\lambda t}\, dt$$

$$= -e^{-\lambda t}\Big|_0^x$$

$$= -(e^{-\lambda x} - 1)$$

$$= 1 - e^{-\lambda x}$$

To compare F_X to F_Y we pick $\delta > 0$ such that

$$\boxed{e^{-\lambda\delta} = 1-p} \quad (\text{i.e. } p = 1-e^{-\lambda\delta})$$

$$\therefore \quad -\lambda\delta = \ln(1-p)$$

$$\therefore \quad \delta = \frac{-\ln(1-p)}{\lambda}.$$

For this value of δ, we have

$$(e^{-\lambda\delta})^n = (1-p)^n$$

$$1-e^{-\lambda\delta n} = 1-(1-p)^n$$

$$\therefore \quad F_{exp}(\underbrace{\delta n}_{\substack{\uparrow \\ \text{Replaces } x}}) = F_{geo}(n)$$

3.3 Normal Random Variables

Let X a cont. r.v. with PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad (x\in\mathbb{R})$$

we call X Normal or Gaussian.

It can be shown

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

so this is a valid PDF.

(see Prob. # 14 in ch. 3, or handout)

we can also show

$$E[X] = \mu$$

$$Var(X) = \sigma^2$$

$$\sqrt{Var(X)} = \sigma$$

It's evident that $f_X(x)$ is symmetric about the line $x = \mu$, since it is invariant under the transformation

$$(x-\mu) \leftrightarrow -(x-\mu).$$

This implies that $E[X] = \mu$.

<u>Proof</u>: exercise.

Lemma

Let X be a cont. r.v. with PDF $f_X(x)$ satisfying

$$f_X(x) = f_X(-x) \quad (\text{for } x \in \mathbb{R}).$$

Then $E[X] = 0$.

Proof

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \underbrace{\int_{-\infty}^{0} x f_X(x)\, dx}_{\substack{t=-x,\ dt=-dx \\ x=-t,\ dx=-dt \\ x \le 0 \Leftrightarrow t \ge 0}} + \int_{0}^{\infty} x f_X(x)\, dx$$

$$= \int_{\infty}^{0} (-t) f_X(-t)(-1)\,dt + \int_{0}^{\infty} x f_X(x)\,dx$$

$$= \int_{\infty}^{0} t f_X(t)\,dt + \int_{0}^{\infty} x f_X(x)\,dx$$

$$= -\int_{0}^{\infty} t f_X(t)\,dt + \int_{0}^{\infty} x f_X(x)\,dx$$

$$= 0$$

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