cse 107 10-30-25

Ex. continuous uniform on $[a,b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a} \cdot \frac{x^2}{2}\Big|_a^b$$

$$= \frac{1}{2}\left(\frac{1}{b-a}\right)(b^2-a^2) = \frac{1}{2}\frac{\cancel{(b-a)}(b+a)}{\cancel{(b-a)}}$$

$$= \boxed{\frac{a+b}{2}}$$

$$E[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \left. \frac{x^3}{3} \right|_a^b$$

$$= \frac{1}{3} \frac{b^3 - a^3}{b-a} = \frac{1}{3} \frac{\cancel{(b-a)}(b^2+ab+a^2)}{\cancel{(b-a)}}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4(b^2+ab+a^2) - 3(a^2+2ab+b^2)}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \boxed{\frac{(b-a)^2}{12}}$$

Ex. recall Alice's driving time

$$f_X(x) = \begin{cases} 1/6 & 40 \le x < 44 \\ 1/18 & 44 \le x \le 50 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{40}^{44} \frac{1}{6} x\,dx + \int_{44}^{50} \frac{1}{18} x\,dx$$

$$= \frac{1}{6} \cdot \frac{1}{2} \left(44^2 - 40^2\right) + \frac{1}{18} \cdot \frac{1}{2} \left(50^2 - 44^2\right)$$

$$= \text{--------} = \boxed{\frac{131}{3}}$$ check ✓

<u>check</u>:

$$E[X^2] = \frac{17{,}228}{9}$$ check ?

$$Var(X) = \boxed{\frac{67}{9}}$$ check ?

Ex Exponential r.v.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$ is a parameter.

check

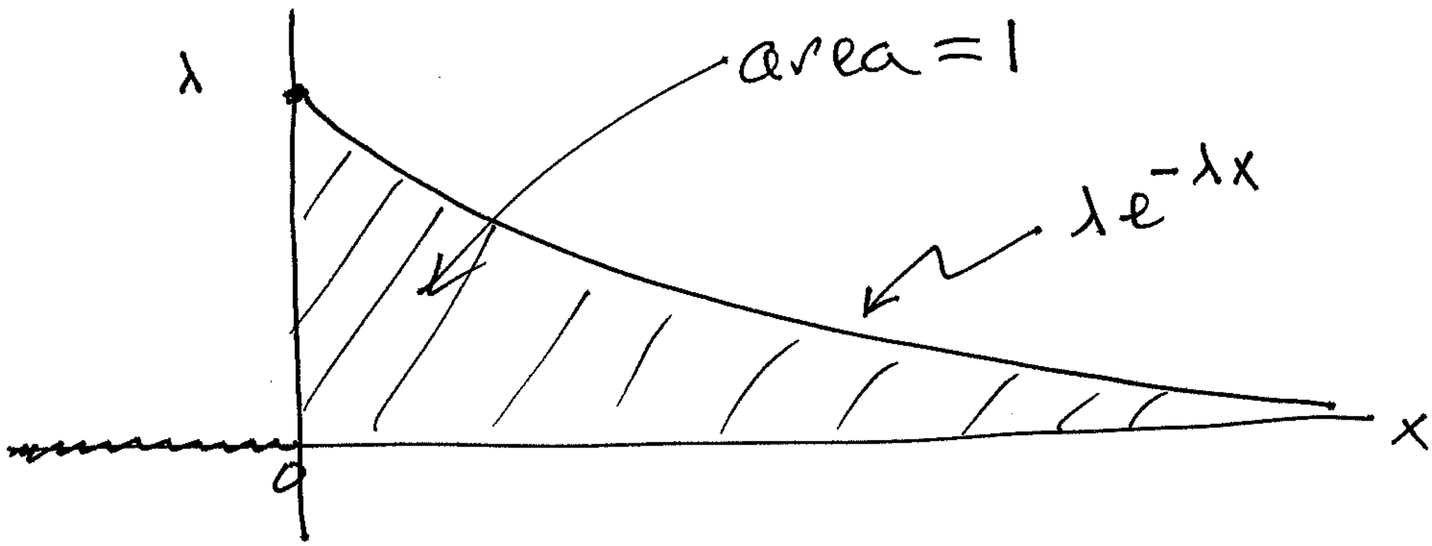
$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= -\int_0^{\infty} (-\lambda) e^{-\lambda x}\,dx$$

$$= -e^{-\lambda x}\Big|_0^{\infty}$$

$$= -\lim_{a\to\infty}\left(e^{-\lambda a} - 1\right)$$

$$= -(0-1) = \boxed{1}$$



show: $E[X] = \boxed{\frac{1}{\lambda}}$

$$E[X^2] = \frac{2}{\lambda^2}$$

$$Var(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda^2}\right) = \boxed{\frac{1}{\lambda^2}}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= -\int_0^{\infty} x\left(-\lambda e^{-\lambda x}\right)dx$$

$$u = x \qquad dv = -\lambda e^{-\lambda x}\,dx$$

$$du = dx \qquad v = e^{-\lambda x}$$

$$= -\left(x e^{-\lambda x}\Big|_0^{\infty} - \int_0^{\infty} e^{-\lambda x}\,dx \right)$$

$$= -\left((0-0) - \left(-\frac{1}{\lambda}\right) e^{-\lambda x}\Big|_0^{\infty} \right)$$

$$= -\left(\frac{1}{\lambda}(0-1) \right) = \boxed{\frac{1}{\lambda}}$$

$$E[X^2] = -\int_0^\infty x^2(-\lambda e^{-\lambda x})\,dx$$

$$u = x^2 \quad dv = -\lambda e^{-\lambda x}dx$$

$$du = 2x\,dx \quad v = e^{-\lambda x}$$

$$= -\left(x^2 e^{-\lambda x}\Big|_0^\infty - \frac{2}{\lambda}\int_0^\infty x(\lambda e^{-\lambda x})\,dx \right)$$

$$= -\left(0 - \frac{2}{\lambda}\cdot\frac{1}{\lambda} \right) = \frac{2}{\lambda^2}$$

$$\therefore \ Var(X) = \frac{1}{\lambda^2}.$$

$E[X]$ has units $\dfrac{\text{time}}{\text{event}}$

$\dfrac{1}{\lambda}$ " " $\dfrac{\text{time}}{\text{event}}$

λ " " $\dfrac{\text{events}}{\text{time}}$

so λ is the "event rate"

Lemma

$$P(X \geq a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \\ 1 & \text{if } a \leq 0 \end{cases}$$

Proof

Let $a > 0$. Then

$$P(X \geq a) = -\int_a^\infty -\lambda e^{-\lambda x}\,dx$$

$$= -\left(e^{-\lambda x}\right)\Big|_a^\infty$$

$$= -\left(0 - e^{-\lambda a}\right)$$

$$= e^{-\lambda a}.$$

$\blacksquare$

Ex.

The time until the next earthquake (4.0 or above) is exponential with mean 53 minutes (world-wide).

What is the probability that such an earthquake happens in next 10 minutes?

solution

$$E[X] = \frac{1}{\lambda} = 53 \Rightarrow \lambda = \frac{1}{53}$$

hence

$$P(0 \le X \le 10) = P(X \ge 0) - P(X > 10)$$

$$= 1 - e^{-\frac{10}{53}} = \boxed{.1719}$$

<u>Exercise</u>

determine a such that the Probability of an earthquake in next a minutes is 0.99

<u>answer</u> $a = 53 \ln(100) = \boxed{244.07 \text{ min}}$

$= \boxed{4.068 \text{ hrs}}$

3.2 Cumulative Distribution Functions

(CDFs)

Defn

Let X be a random variable.

The CDF of X is

$$\boxed{F_X(x) = P(X \le x)}$$

- if X is discrete

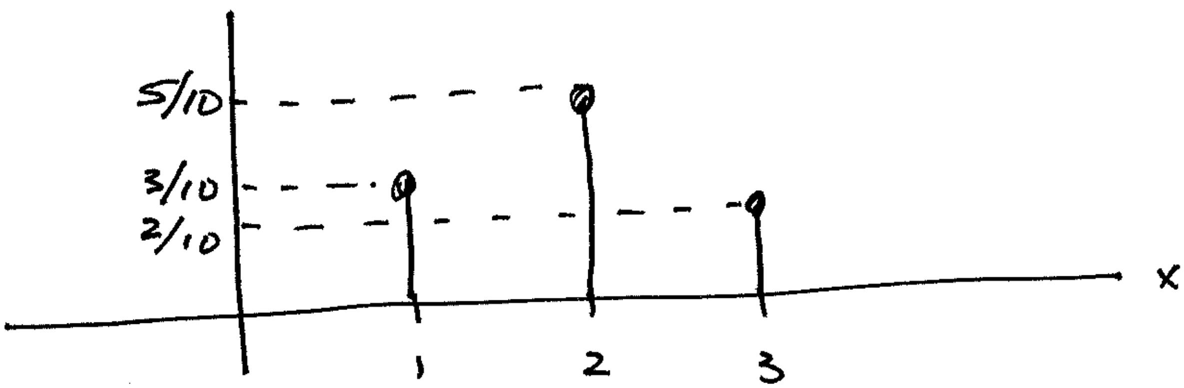
$$F_X(x) = \underbrace{\sum_{k \le x}}_{\text{sum over range}(X) \cap (-\infty, x]} p_X(k)$$

- if X is continuous

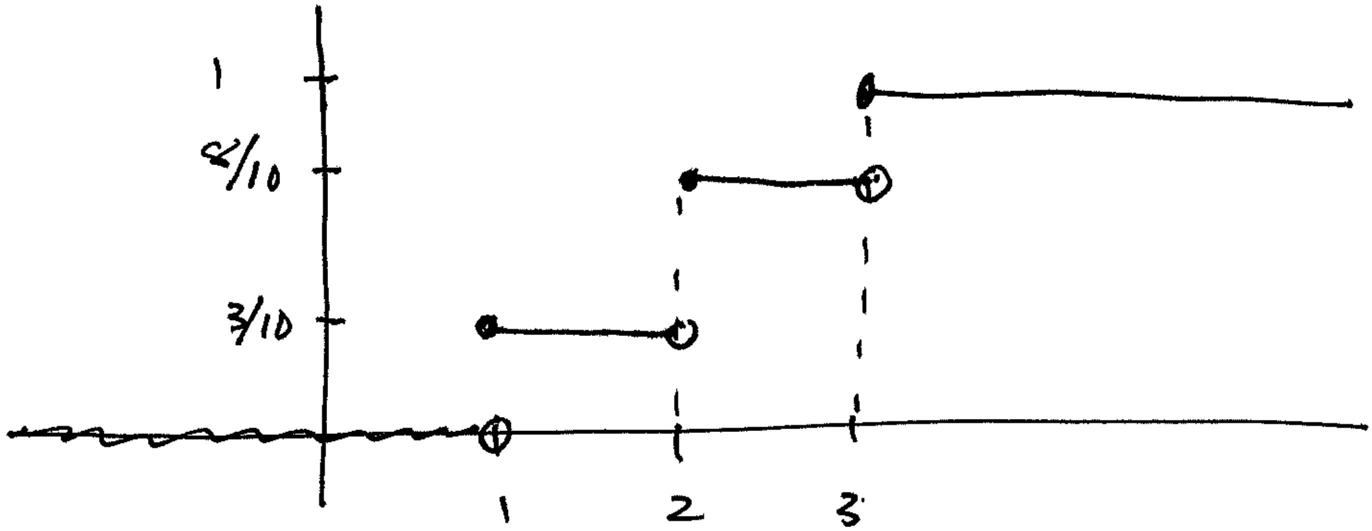
$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

Ex.

PMF: $$P_X(k) = \begin{cases} 3/10 & k=1 \\ 5/10 & k=2 \\ 2/10 & k=3 \\ 0 & \text{otherwise} \end{cases}$$

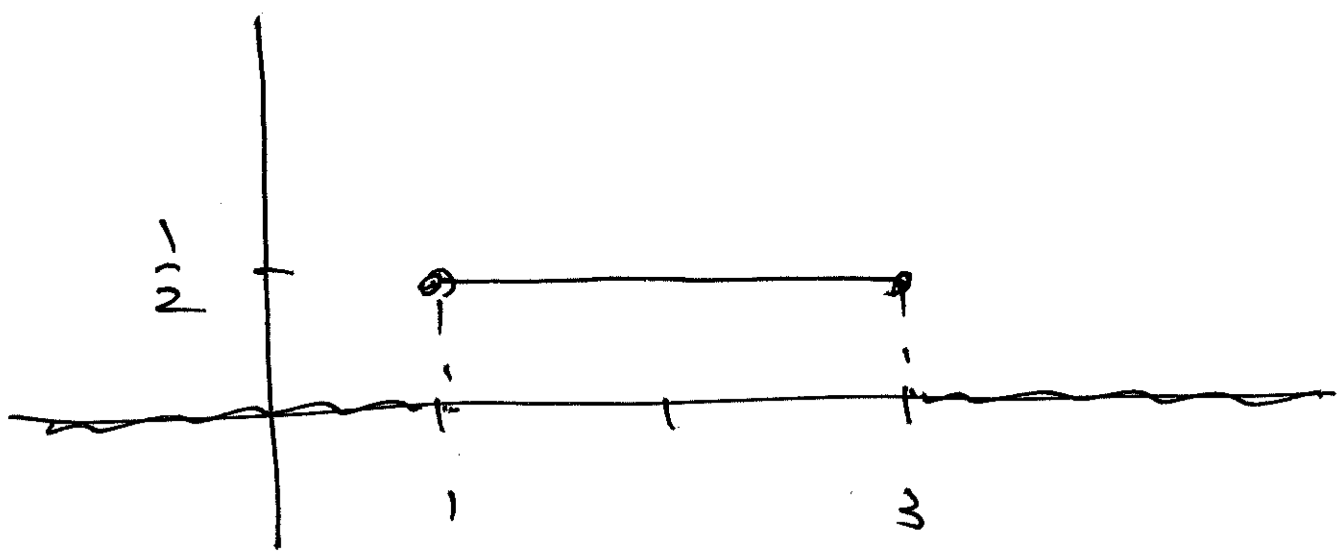


CDF: $$F_X(x) = \begin{cases} 0 & x < 1 \\ 3/10 & 1 \le x < 2 \\ 8/10 & 2 \le x < 3 \\ 1 & 3 \le x \end{cases}$$



Ex. Cont. Unif on [1, 3]

PDF: $f_X(x) = \begin{cases} \frac{1}{2} & 1 \le x \le 3 \\ 0 & \text{other} \end{cases}$



CDF: $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

(say $1 \le x \le 3$)

$$= \int_{1}^{x} \tfrac{1}{2}\,dt$$

$$= \tfrac{1}{2}\,t \Big|_{1}^{x} = \tfrac{1}{2}(x-1)$$

$$= \frac{x-1}{2}$$

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \dfrac{x-1}{2} & 1 \le x \le 3 \\ 1 & x > 3 \end{cases}$$