CSE 107 10-3-25

Supplemental Lecture

Total Probability Theorem

Let $A_1, \dots, A_n$ form a partition of Ω, and assume $P(A_i) > 0$ for $i = 1, \dots, n$. Let $B \subseteq \Omega$. Then

$$\boxed{P(B) = P(A_1) \cdot P(B|A_1) + \dots + P(A_n) \cdot P(B|A_n)}$$

Ex. Roll a 6-sided fair die. If the result $\in \{1, 2, 3\}$, roll again, then stop. If 1st roll $\in \{4, 5, 6\}$, no second roll.

$$\text{Sum} = \begin{cases} 1^{st}\text{ roll} + 2^{nd}\text{ roll} & \text{if } 1^{st} \in \{1,2,3\} \\ 1^{st}\text{ roll} & \text{if } 1^{st} \in \{4,5,6\} \end{cases}$$

Find $P(\text{sum} \geq 5)$.

Let $A_i = \{1^{st} \text{ roll is } i\}$ $(1 \leq i \leq 6)$.

Then $P(A_i) = \frac{1}{6}$ $(1 \leq i \leq 6)$. Let

$B = \{\text{sum} \geq 5\}$.

<u>Note</u>:

if A_1, then $B \iff 2^{nd}$ roll $\in \{4, 5, 6\}$

if A_2, then $B \iff 2^{nd}$ roll $\in \{3, 4, 5, 6\}$

if A_3, then $B \iff 2^{nd}$ roll $\in \{2, 3, 4, 5, 6\}$

Thus

$$P(B|A_1) = \frac{3}{6} = \frac{1}{2}$$

$$P(B|A_2) = \frac{4}{6} = \frac{2}{3}$$

$$P(B|A_3) = 5/6$$

$$P(B|A_4) = 0$$

$$P(B|A_5) = 1$$

$$P(B|A_6) = 1$$

Hence

$$P(B) = P(B|A_1)\cdot P(A_1) + \cdots + P(B|A_6)\cdot P(A_6)$$

$$= \frac{1}{2}\cdot\frac{1}{6} + \frac{2}{3}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{1}{6} + 0\cdot\frac{1}{6} + 1\cdot\frac{1}{6} + 1\cdot\frac{1}{6}$$

$$= \frac{1}{6}\left(\frac{1}{2} + \frac{2}{3} + \frac{5}{6} + 0 + 1 + 1\right)$$

$$= \frac{1}{6}\left(\frac{24}{6}\right) = \frac{4}{6} = \boxed{\frac{2}{3}}$$

<u>Another way</u>:

$$\Omega \left\{ \begin{array}{cccccc} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 4 & & & & & \\ 5 & & & & & \\ 6 & & & & & \end{array} \right. \longleftarrow B = \{\text{sum} \geq 5\}$$

$$P(B) = \frac{12}{36} + \frac{2}{6} = \frac{1}{3} + \frac{1}{3} = \boxed{\frac{2}{3}}$$

Baye's Rule

Recall $P(A \cap B) = P(B) \cdot P(A|B)$
$$\begin{aligned} &\quad \parallel \\ P(B \cap A) &= P(A) \cdot P(B|A) \end{aligned}$$

$\therefore \quad P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$

$$\therefore \quad \boxed{P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}}$$

This simplest form of Baye's rule

Let $A_1, \ldots, A_n$ be a partition of Ω and $P(A_i) > 0$. Assume $P(B) > 0$. Then for any $i = 1, \ldots n$, we have

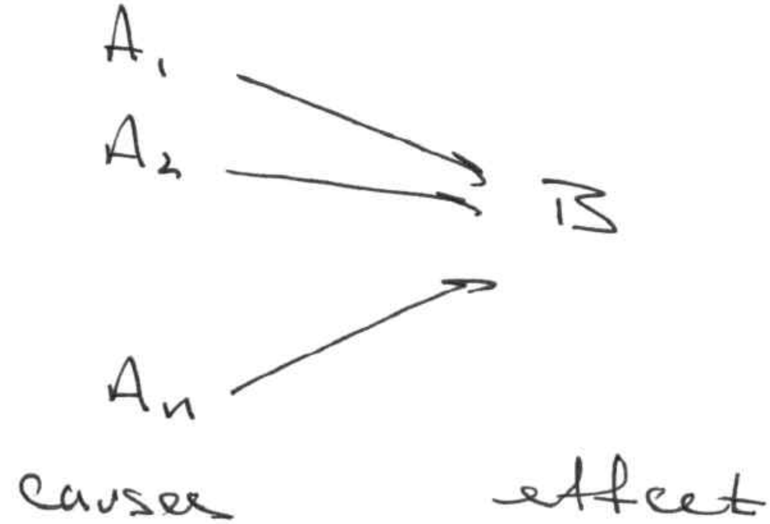
$$\boxed{P(A_i|B) = \frac{P(A_i)\cdot P(B|A_i)}{P(A_1)\cdot P(B|A_1) + \cdots + P(A_n)\cdot P(B|A_n)}}$$

This is called <u>Baye's Rule</u>.

Baye's is used for 'inference' Problems.
We have causes:

$$A_1, A_2, \ldots, A_n$$

that may Produce an effect B.



<u>note</u> $P(B|A_i)$ models strength of causation

$$A_i \xrightarrow[\text{causation}]{} B$$

$$P(B|A_i)$$

often, we wish to go from effect to cause. $P(A_i|B)$ models the strengh of the inference of A_i from B.

$$A_i \xleftarrow[\text{inference}]{} B$$

$$P(A_i|B)$$

prior

likelihood

$$\underbrace{P(A_i|B)}_{\uparrow} = \frac{\overbrace{P(A_i)}^{\text{prior}}\ \overbrace{P(B|A_i)}^{\text{likelihood}}}{\underbrace{P(B)}_{\text{marginal}}}$$

Posterior

marginal

Ex Disease example again

$T = \{\text{test positive}\}$

$D = \{\text{has disease}\}$

Given information:

$$P(T|D) = .99$$

$$P(T|D^c) = .10$$

$$P(D) = .05 \ , \ P(D^c) = .95$$

find P(disease|test), i.e

$$P(D|T) = \frac{P(D)\cdot P(T|D)}{P(T)}$$

$$= \frac{P(D)\cdot P(T|D)}{P(D)\cdot P(T|D) + P(D^c)\cdot P(T|D^c)}$$

$$= \frac{(.05)(.99)}{(.05)(.99)+(.95)(.10)} = .34256 \approx \boxed{.3426}$$

round to 4 digits

hw1 #2

Players ranked: $1 < 2 < 3$

$\boxed{\text{Assume play in order } 1, 2, 3}$

$$\left.\begin{aligned} A &= \{\text{win against } 1\},\ P(A) = P_1 \\ B &= \{\text{" \quad " } 2\},\ P(B) = P_2 \\ C &= \{\text{" \quad " } 3\},\ P(C) = P_3 \end{aligned}\right\} P_1 > P_2 > P_3$$

$$E = \{\text{win tournament}\}$$

$$= (A \cap B) \cup (B \cap C)$$

need

$$P(A \cap B) = P(A) \cdot \overbrace{P(B|A)}^{\text{assume} = P(B)} = P(A) \cdot P(B) = P_1 \cdot P_2$$

$$P(B \cap C) = \ldots \quad \ldots = P_2 \cdot P_3$$

$$P(A \cap B \cap C) = \ldots \quad \ldots = P_2 \cdot P_2 \cdot P_3$$

1.5 Independence

It's possible that knowledge of $B \subseteq \Omega$ tells us nothing about $A \subseteq \Omega$. i.e.

$$* \qquad P(A|B) = P(A)$$

In this case we say A is independent of B. Recall

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Thus * gives

$$** \qquad \boxed{P(A \cap B) = P(A) \cdot P(B)}$$

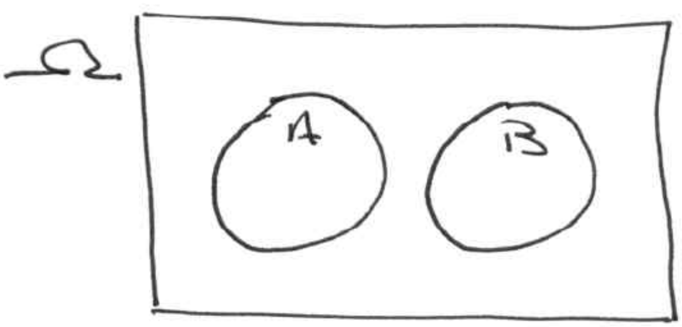
<u>Defn</u>
events A, B are <u>independent</u> iff ** holds.

Independence has a simple intuition:
A, B represent physically non-interacting processes.

Independence is not easily visualized as a Venn diagram

<u>note</u> $A \cap B = \phi \not\Rightarrow A, B$ independent

why? $A \cap B = \phi \Rightarrow P(A \cap B) = 0$. But neither $P(A)$ nor $P(B)$ need be impossible



$A \cap B = \phi$

$P(A) > 0$
$P(B) > 0$
$\therefore P(A) \cdot P(B) > 0$
but
$P(A \cap B) = 0$

Ex. Throw 2 fair dice. let

$A_i = \{ 1^{st} \text{ die is } i \}$ $\quad 1 \le i \le 6$

$B_j = \{ 2^{nd} \text{ die is } j \}$ $\quad 1 \le j \le 6$

fair means $P(A_i) = \frac{1}{6}$, $P(B_j) = \frac{1}{6}$

Assume A_i, B_j are independent.

Then

$$P((i,j)) = P(A_i \cap B_j) = P(A_i) \cdot P(B_j) = \frac{1}{36}$$

which is the discrete uniform probability law.

Let $C = \{ \text{sum} = 5 \}$

$= \{ 14, 23, 32, 41 \}$

$\therefore P(C) = \frac{4}{36} = \frac{1}{9}$

Is C independent of A_2?

$$A_2 = \{21, 22, 23, 24, 25, 26\}$$

$\therefore \quad A_2 \cap C = \{23\}$

$\therefore \quad P(A_2 \cap C) = \frac{1}{36}$

But

$$P(A_2) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{9} = \frac{1}{54} \neq \frac{1}{36}$$

$\therefore \quad A_2, C$ <u>are not</u> independent.

Let $D = \{\text{doubles}\} = \{11, 22, 33, 44, 55, 66\}$

$\therefore \quad P(D) = \frac{6}{36} = \frac{1}{6}$

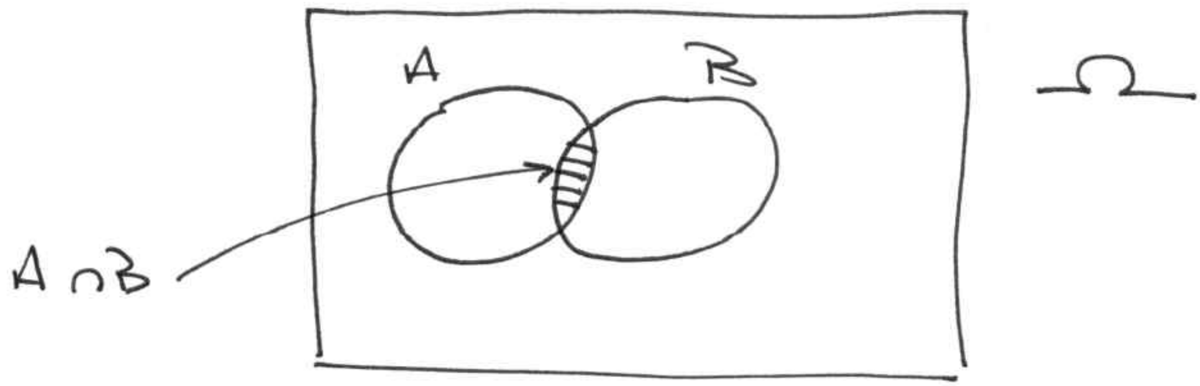
$D \cap A_2 = \{22\}$

$\therefore \quad P(D \cap A_2) = \frac{1}{36}$

But $P(D) \cdot P(A_2) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = P(D \cap A_2)$

Thus A_2, D are independent. ∎

Assume Probability is normalized area, as in a continuous model



$$P(A|B) = P(A)$$

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

$$\frac{\dfrac{\text{area}(A\cap B)}{\cancel{\text{area}(\Omega)}}}{\dfrac{\text{area}(B)}{\cancel{\text{area}(\Omega)}}} = \frac{\text{area}(A)}{\text{area}(\Omega)}$$

$$\boxed{\begin{array}{l}\text{assume}\\ \text{area}(\Omega)=1\end{array}}$$

$$\therefore \quad \frac{\text{area}(A\cap B)}{\text{area}(B)} = \text{area}(A)$$