CSE 107 10-28-25

Recall:

$$S_n = \frac{X_1 + \cdots + X_n}{n}$$

Bernoulli(p) indep.

is sample mean, then

$$E[S_n] = p$$

compute $Var(S_n)$:

$$Var(S_n) = Var\left(\sum_{i=1}^{n} \frac{1}{n} \cdot X_i\right)$$

$$= \sum_{i=1}^{n} Var\left(\frac{1}{n} \cdot X_i\right)$$

$$= \sum_{i=1}^{n} \frac{1}{n^2} Var(X_i)$$

$$= \sum_{i=1}^{n} \frac{1}{n^2} \cdot p(1-p)$$

$$= \cancel{n} \cdot \frac{1}{n^{\cancel{2}}} p(1-p)$$

$$= \frac{p(1-p)}{n} \longrightarrow 0 \quad \text{as } n \to \infty$$

<u>Recall</u>: $Var(Y) = 0 \Rightarrow Y = \text{const} = E[Y]$

Note

Even if X_i are not Bernoulli, but still independent, a similar calculation (exercise) gives

$$Var(S_n) = \frac{Var(X)}{n}$$

any one of the X_i

and $E[S_n] = E[X]$

since $Var(S_n) \to 0$ as $n \to \infty$, S_n is a good approx. to mean $E[X]$.

3.1 Continuous Random Variables, PDFs

Defn

A random variable $X : \Omega \to \mathbb{R}$ is called continuous iff there exists a function

$$f_X : \mathbb{R} \to \mathbb{R}$$

called the Probability Density Function (PDF) of X, such that

$$P(X \in I) = \int_I f_X(x)\,dx$$

where $I \subseteq \mathbb{R}$. In particular if $I = [a, b]$, then

$$P(a \le X \le b) = \int_a^b f_X(x)\,dx .$$

<u>note</u> integrating over $[a,b]$, (a,b), $[a,b)$, $(a,b]$ are all the same.

since $P(X=a) = \int_a^a f_X(x)\,dx = 0$

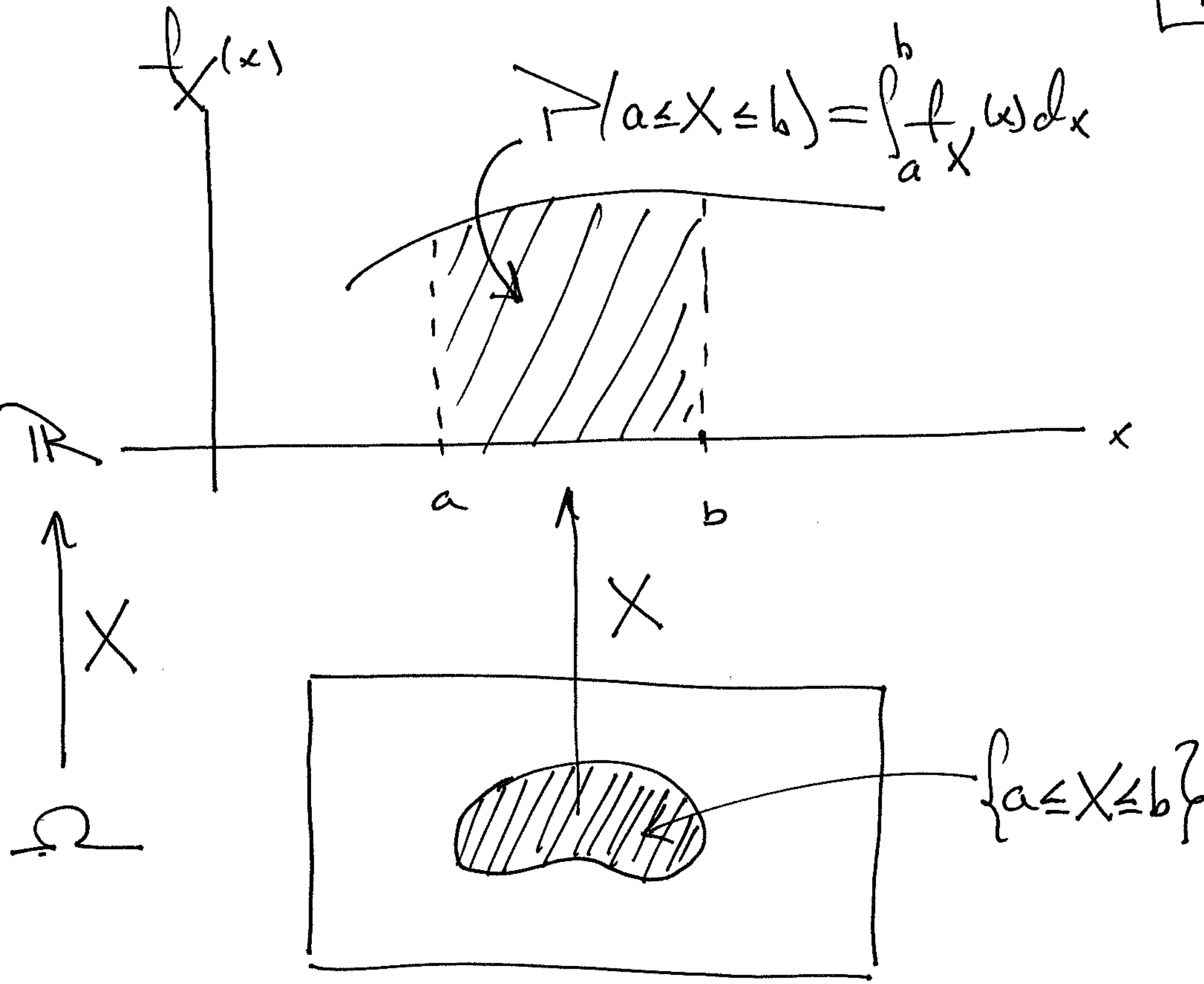
<u>note</u> to be a valid PDF, must have

$$\int_{-\infty}^{\infty} f_X(x)\,dx = P(X \in \mathbb{R})$$

$$= P(\Omega)$$

$$= 1$$

Also, must have $f_X(x) \geq 0$ for all $x \in \mathbb{R}$.



<u>note</u>: • PMF gives Probability

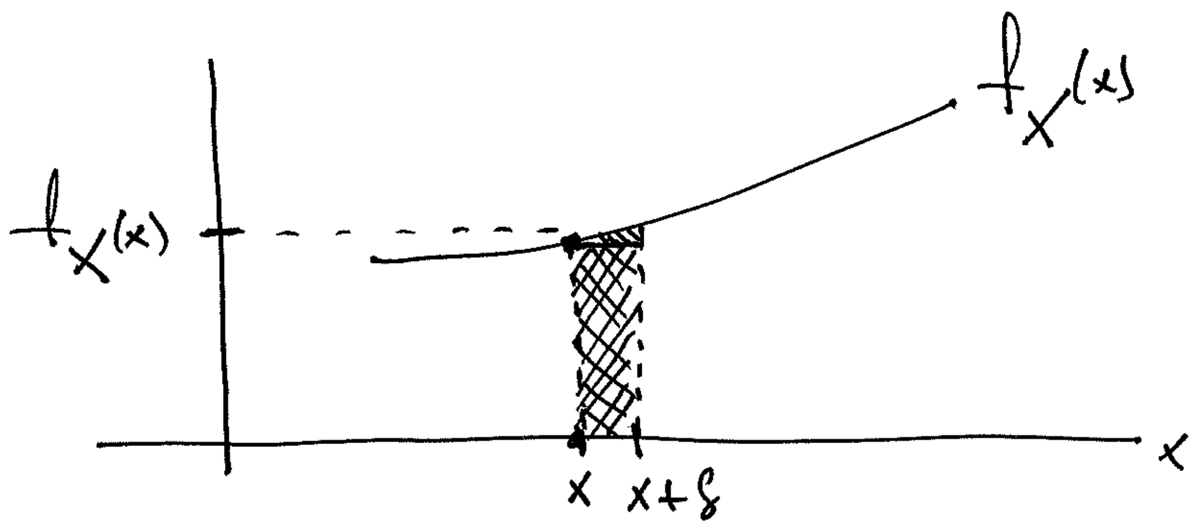
• PDF gives Density

let $\delta > 0$ be a small real number,

suppose $f_X(x)$ is continuous on

interval $[x, x+\delta]$. $\therefore f_X(x)$ is

approx. const. on $[x, x+\delta]$.



$$\underbrace{P(x \le X \le x+\delta)}_{\text{Probability}} = \int_x^{x+\delta} f_X(t)\,dt \approx f_X(x) \cdot \underbrace{\delta}_{\text{length}}$$

$\therefore f_X(x)$ is $\frac{\text{Prob.}}{\text{len.}}$ i.e. density.

<u>note</u> PDF may have jump discontinuities.

<u>Ex</u>. Continuous uniform r.v. on $[a, b]$

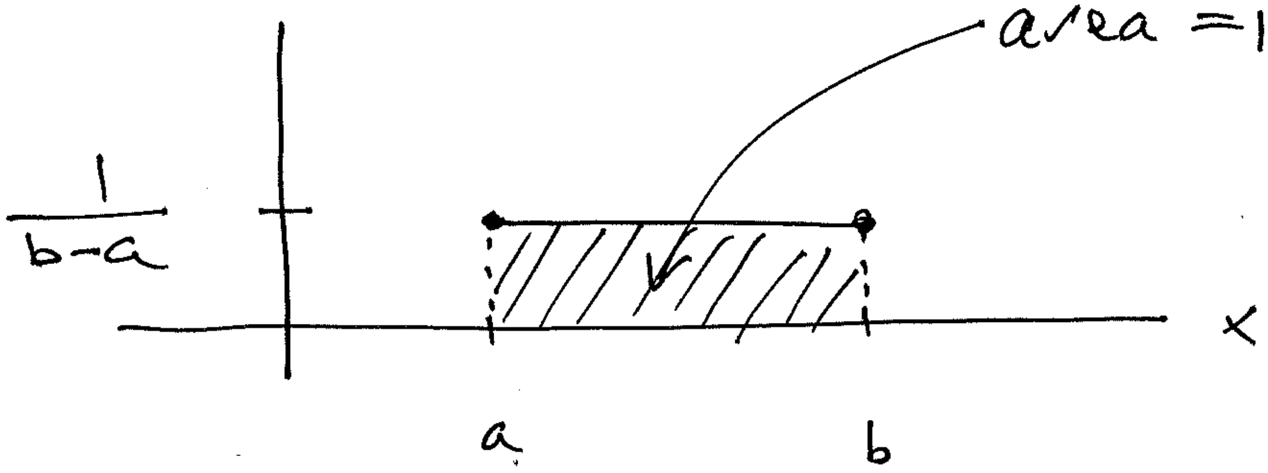
$$f_X(x) = \begin{cases} c & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

To determine c, use fact

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_a^b c\,dx$$

$$= cx\Big|_a^b = c(b-a) \quad \therefore \boxed{c = \frac{1}{b-a}}$$

Thus



So

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Ex Piecewise constant PDF

Alice's driving time to work is 40-44 minutes in good weather, and 44-50 minutes in bad.

Suppose $P(\text{good weather}) = \frac{2}{3}$

Find PDF of X, Alice's driving time.

driving time is uniform in each case.

Soln

We're given

$$f_X(x) = \begin{cases} c_1 & \text{if } 40 \leq x \leq 44 \text{ (good)} \\ c_2 & \text{if } 44 \leq x \leq 50 \text{ (bad)} \\ 0 & \text{otherwise} \end{cases}$$

Also

$$\frac{2}{3} = P(\text{good}) = \int_{40}^{44} c_1 \, dx = c_1(40-44) = 4c_1$$

$$\therefore \quad c_1 = \frac{2}{12} = \boxed{\frac{1}{6}}$$

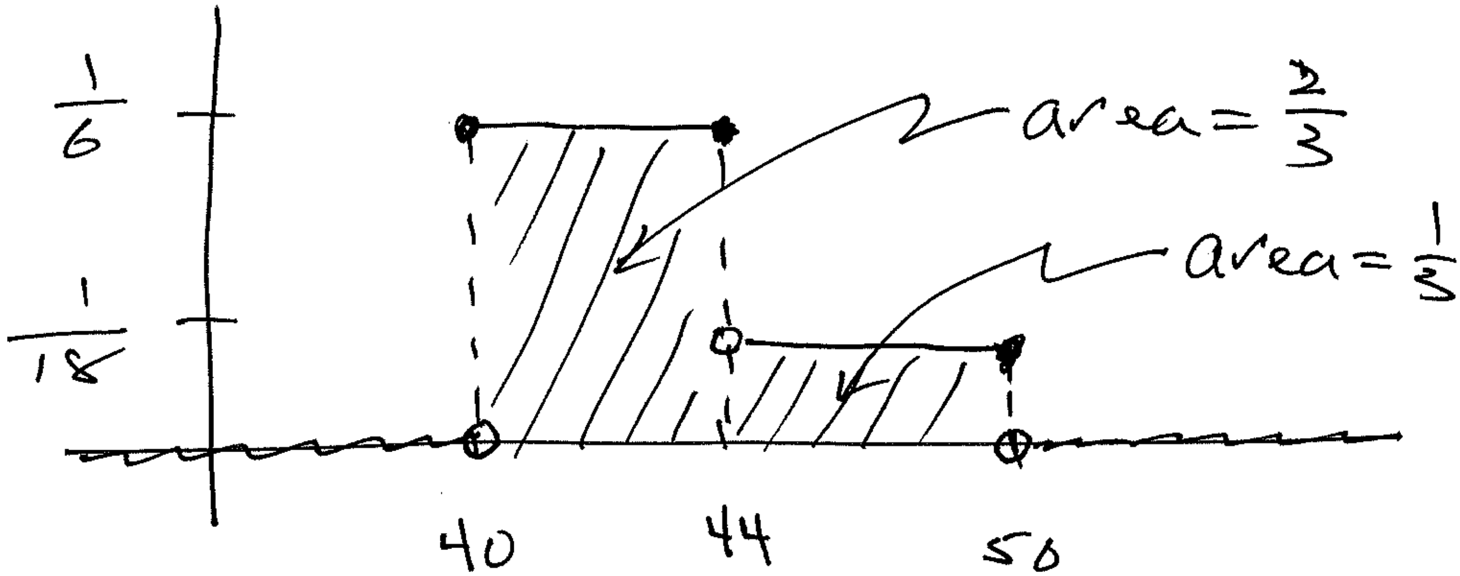
Also

$$\frac{1}{3} = 1 - P(\text{good}) = P(\text{bad})$$

$$= \int_{44}^{50} c_2 \, dx = 6 c_2$$

$$\therefore \boxed{c_2 = \frac{1}{18}}$$

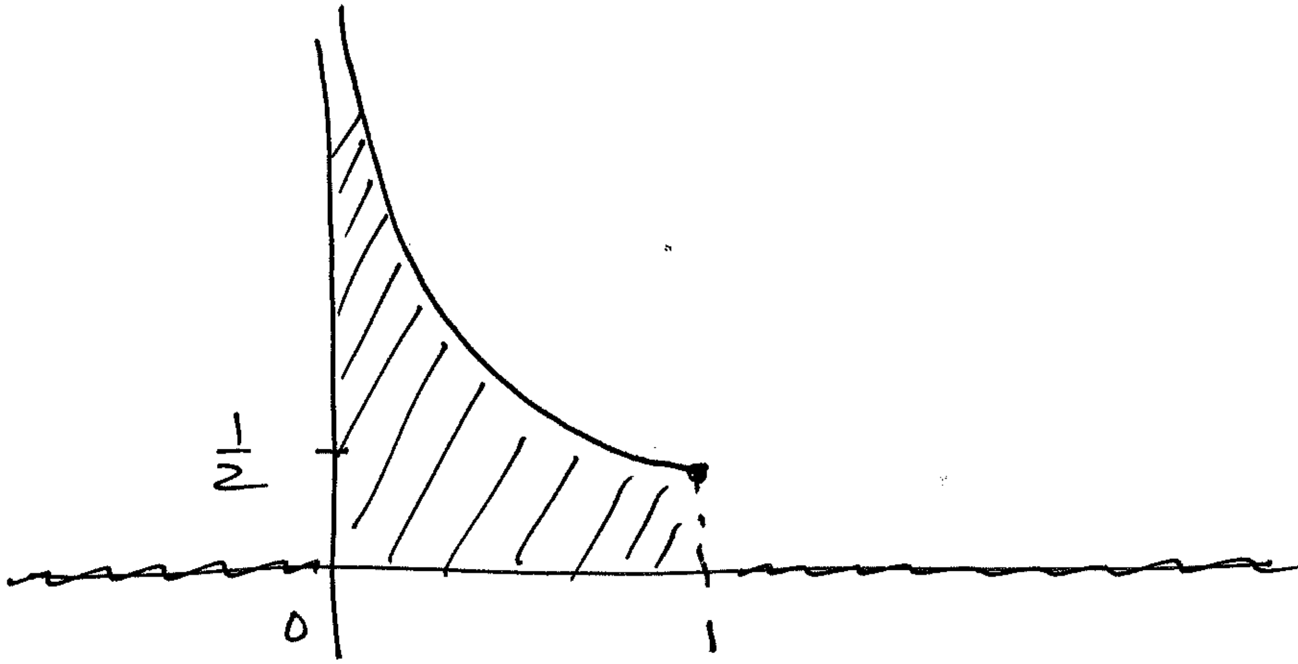
$$\therefore \quad f_X(x) = \begin{cases} \frac{1}{6} & \text{if } 40 \le x \le 44 \\ \frac{1}{18} & \text{if } 44 < x \le 50 \\ 0 & \text{otherwise} \end{cases}$$



<u>note</u> a PDF need not be bounded above.

<u>EX</u>

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\int_0^1 \frac{1}{2\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{2}x^{-\frac{1}{2}}\,dx$$

$$= \lim_{a\to 0^+} \cancel{\tfrac{1}{2}}\,\frac{x^{1/2}}{\cancel{\tfrac{1}{2}}}\Bigg|_a^1 = \lim_{a\to 0^+}\left(1-\sqrt{a}\right)$$

$$= 1 - 0 = \boxed{1}$$

Defn

The Expected Value $E[X]$ of a cont. r.v. X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

Expected Value rule

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$$

[See problem #4 on p. 185 with solution]

The n^{th} moment of X is

$$E[X^n].$$

The variance of X is

$$Var(X) = E[(X - E[X])^2]$$

$$= E[X^2] - E[X]^2$$

exercise! Prove this!

Proof

let $\mu = E[X]$

$$\therefore Var(X) = E[(X-\mu)^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - 2\mu \int_{-\infty}^{\infty} x f_X(x)\,dx + \mu^2 \int_{-\infty}^{\infty} f_X(x)\,dx$$

$$= E[X^2] - 2\mu \underbrace{E[X]}_{\mu} + \mu^2$$

$$= E[X^2] - \mu^2$$

$$= E[X^2] - E[X]^2 \quad \blacksquare$$

If $Y = aX + b$, then

- $E[Y] = a\,E[X] + b$

and

- $\mathrm{Var}(Y) = a^2\,\mathrm{Var}(X)$

<u>Exercise:</u> Prove these.