

CSE 107 10-21-25

- Midterm exam 1 : Thur. 10/23/25
 - exam: 1:30 - 2:35
 - break: 2:35 - 2:40
 - lecture: 2:40 - 3:05
- hw3 is due tonight 10:00 PM

2.7 Independence

L2

Defn

We say r.v.s X, Y are
independent iff for all
 $x \in \text{range}(X)$ and $y \in \text{range}(Y)$:

$$\{X=x\}, \{Y=y\}$$

are independent events, i.e.

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

equivalently

$$P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y)$$

if $P_Y(y) > 0$ for $y \in \text{range}(Y)$, this
is equiv. to

$$\frac{P_{X,Y}(x,y)}{P_Y(y)} = P_X(x)$$

i.e.

$$P_{X|Y}(x|y) = P_X(x)$$

let $P(A) > 0$, $A \subseteq \Omega$.

Then $P(\cdot | A)$ is another
Prob. law.

Defn

We say r.v.'s X, Y are
conditionally independent given A

iff

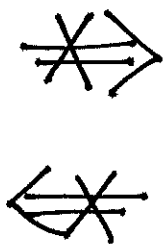
$$P(X=x, Y=y | A) = P(X=x | A) \cdot P(Y=y | A)$$

ie.

$$P_{X,Y|A}^{(x,y)} = P_{X|A}^{(x)} \cdot P_{Y|A}^{(y)}$$

note :

conditional
independence
of X, Y



independence
of X, Y

See fig. 2.15 on p. 111 of text.
for an example

Theorem

If X, Y are independent r.v.s,

then

$$E[X \cdot Y] = E[X] \cdot E[Y]$$

Proof

$$E[X \cdot Y] = \sum_x \sum_y x \cdot y P_{X,Y}(x,y)$$

$$= \sum_x \left(\sum_y x y P_X(x) P_Y(y) \right)$$

$$= \sum_x \left(x P_X(x) \sum_y y P_Y(y) \right)$$

□

$$= \left(\sum_x x P_X(x) \right) \left(\sum_y y P_Y(y) \right)$$

$$= E[X] \cdot E[Y]$$

■

Remark

if X, Y are indep., then so are $g(X)$ and $h(Y)$.

(see #44 on p. 134). Thus

$$E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]$$

Theorem

Let X, Y are indep. v. v. s, then

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

Proof

Let $U = X - E[X], V = Y - E[Y]$.

Then

$$E[U] = E[X] - E[X] = 0$$

and

$$E[V] = \dots \dots = 0$$

Also

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$$\begin{aligned}u+v &= X - E[X] + Y - E[Y] \\&= (X+Y) - E[X+Y]\end{aligned}$$

Thus

$$\text{Var}(X+Y) = E\left[\left((X+Y) - E[X+Y]\right)^2\right]$$

$$= E[(u+v)^2]$$

$$= E[u^2 + 2uv + v^2]$$

$$= E[u^2] + 2E[u \cdot v] + E[v^2]$$

$$= \text{Var}(X) + 2 \underbrace{E[U]}_0 \cdot \underbrace{E[V]}_0 + \text{Var}(Y)$$

$$= \text{Var}(X) + \text{Var}(Y).$$

