CSE 107 10-21-25

- Midterm exam 1 : Thur. 10/23/25

  exam: 1:30 - 2:35

  break: 2:35 - 2:40

  lecture: 2:40 - 3:05

- hw3 is due tonight 10:00 PM

2.7 Independence

Defn

We say r.v.s X, Y are independent iff for all $x \in range(X)$ and $y \in range(Y)$:

$$\{X = x\}, \{Y = y\}$$

are independent events, i.e.

$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$$

equivalently

$$\boxed{p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y)}$$

if $p_Y(y) > 0$ for $y \in \underline{\text{range}}(Y)$, this is equiv. to

$$\frac{p_{X,Y}(x,y)}{p_Y(y)} = p_X(x)$$

i.e.

$$p_{X|Y}(x|y) = p_X(x)$$

let $P(A) > 0$, $A \subseteq \Omega$.

Then $P(\cdot \mid A)$ is another Prob. law.

<u>Defn</u>

We say r.v.s X, Y are <u>conditionally independent given A</u>

iff

$$P(X=x, Y=y \mid A) = P(X=x \mid A) \cdot P(Y=y \mid A)$$

i.e.

$$\boxed{P_{X,Y|A}(x,y) = P_{X|A}(x) \cdot P_{Y|A}(y)}$$

note:

$$\boxed{\text{conditional independence of } X, Y} \;\; \begin{array}{c} \not\Longrightarrow \\ \not\Longleftarrow \end{array} \;\; \boxed{\text{independence of } X, Y}$$

See fig. 2.15 on p. 111 of text for an example

Theorem

If X, Y are independent r.v.s, then

$$E[X \cdot Y] = E[X] \cdot E[Y]$$

Proof

$$E[X \cdot Y] = \sum_x \sum_y x \cdot y \, P_{X,Y}(x,y)$$

$$= \sum_x \left(\sum_y x y \, P_X(x) P_Y(y) \right)$$

$$= \sum_x \left(x P_X(x) \sum_y y P_Y(y) \right)$$

$$= \left(\sum_x x\, p_X(x)\right)\left(\sum_y y\, p_Y(y)\right)$$

$$= E[X] \cdot E[Y] \quad \blacksquare$$

<u>Rmk</u>

if X, Y are indep., then so are $g(X)$ and $h(Y)$.

(see #44 on P. 134). Thus

$$E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]$$

<u>Theorem</u>

If X, Y are indep. r.v.s, then

$$Var(X+Y) = Var(X) + Var(Y)$$

<u>Proof</u>

let $U = X - E[X]$, $V = Y - E[Y]$.

Then

$$E[U] = E[X] - E[X] = 0$$

and

$$E[V] = \quad \cdots \quad \cdots \quad = 0$$

Also

$$U+V = X - E[X] + Y - E[Y]$$

$$= (X+Y) - E[X+Y]$$

Thus

$$Var(X+Y) = E\left[\left((X+Y) - E[X+Y]\right)^2\right]$$

$$= E\left[(U+V)^2\right]$$

$$= E\left[U^2 + 2UV + V^2\right]$$

$$= E[U^2] + 2E[U \cdot V] + E[V^2]$$

$$= Var(X) + \cancel{2\underbrace{E[U]}_{0}\cdot\underbrace{E[V]}_{0}} + Var(Y)$$

$$= Var(X) + Var(Y). \qquad \blacksquare$$