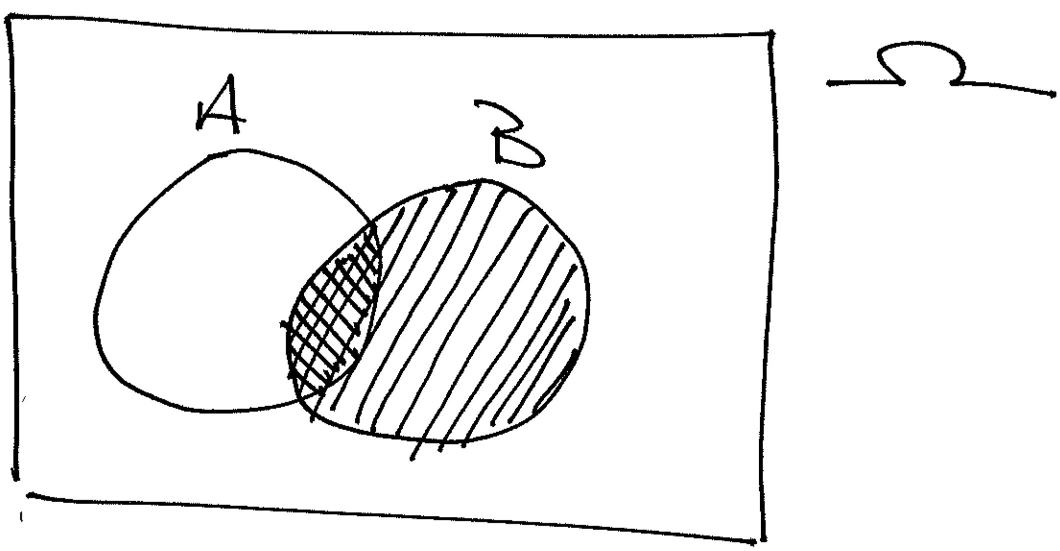
CSE 107 10-2-25

<u>Defn</u> if $P(B) > 0$:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Picture:



<u>note</u>: $P(\cdot \mid B)$ is a valid probability law.

check axioms

(i) $P(A|B) = \dfrac{P(A\cap B)}{P(B)} \geq 0$

(ii) $P(\Omega|B) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

(iii) If $A_1 \cap A_2 = \emptyset$, then

$$P(A_1 \cup A_2 | B) = \frac{P((A_1 \cup A_2)\cap B)}{P(B)}$$

$$= \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1 | B) + P(A_2 | B)$$

For instance,

from:

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

we get:

$$P(A_1 \cup A_2 | B) = P(A_1 | B) + P(A_2 | B) - P(A_1 \cap A_2 | B)$$

<u>note</u> the definition

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Can be written as

$$\boxed{P(A \cap B) = P(A|B) \cdot P(B)}$$

<u>Ex</u> (Disease example)

We have a medical test.

- If subject has disease, then test is positive with Prob. .99

- If subject does not have disease, then test is positive with Prob. .10
- The Prob. that has the disease is .05

Questions

(1) what is Prob. of false positive?

(2) What is Prob. of false negative?

Let

$T = \{\text{test is positive}\}$

$D = \{\text{subject has disease}\}$

We are given:

$$P(T|D) = .99, \quad P(T^c|D) = .01$$

$$P(T|D^c) = .10, \quad P(T^c|D^c) = .90$$

$$P(D) = .05, \quad P(D^c) = .95$$

We seek

$P(T \cap D^c)$ (false positive)

$P(T^c \cap D)$ (false negative)

We have

$$P(T \cap D^c) = P(T \mid D^c) \cdot P(D^c)$$

$$= (.10)(.95) = \boxed{.095}$$

and

$$P(T^c \cap D) = P(T^c \mid D) \cdot P(D)$$

$$= (.01)(.05) = \boxed{.0005}$$

observe:

$$P(A|B)\cdot P(B) = P(A\cap B)$$

$$= P(A)\cdot P(B|A)$$

<u>note</u>:

- $P(A\cap B) = P(A)\cdot P(B|A)$
- $P(A\cap B\cap C) = P(A)\cdot P(B|A)\cdot P(C|A\cap B)$

<u>Proof</u>

$$RHS = \cancel{P(A)}\cdot\frac{\cancel{P(A\cap B)}}{\cancel{P(A)}}\cdot\frac{P(A\cap B\cap C)}{\cancel{P(A\cap B)}}$$

$$= LHS.$$

- $P(A \cap B \cap C \cap D)$

$$= P(A) \cdot P(B|A) \cdot P(C|A \cap B) \cdot P(D|A \cap B \cap C)$$

<u>Exercise</u> Prove this.

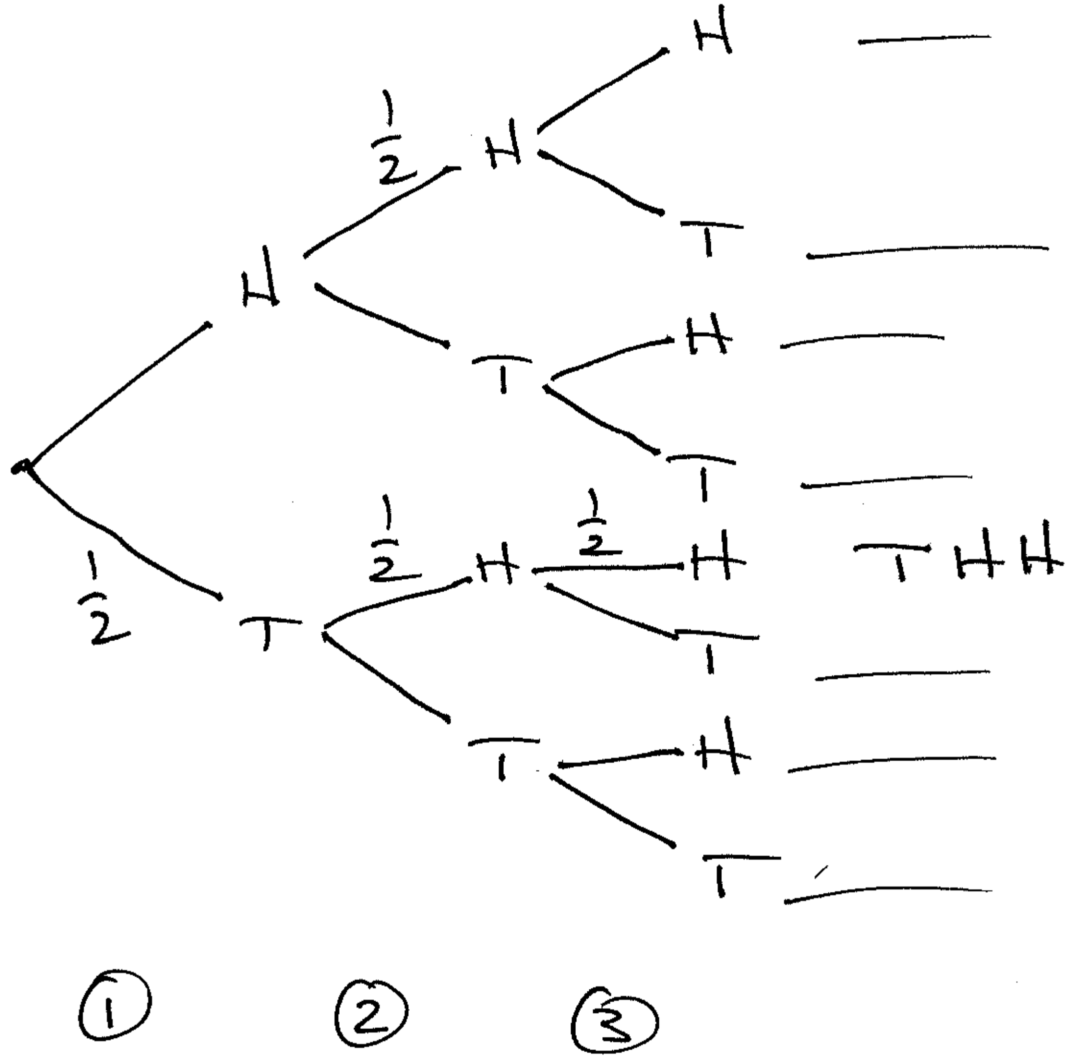
<u>Multiplication rule</u>

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdots\cdots P\left(A_n \middle| \bigcap_{i=1}^{n-1} A_i\right)$$

i.e.

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P\left(A_i \,\middle|\, \bigcap_{j=1}^{i-1} A_j\right)$$

Ex. toss 3 fair coins



Prob

$\frac{1}{8}$

$\frac{1}{8}$

$\vdots$

$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$\vdots$

$\frac{1}{8}$

we recover the discrete uniform probability law.

<u>note</u>: still true (i.e. multiply down branches) even for a weighted coin

<u>Ex.</u> have 3 coins

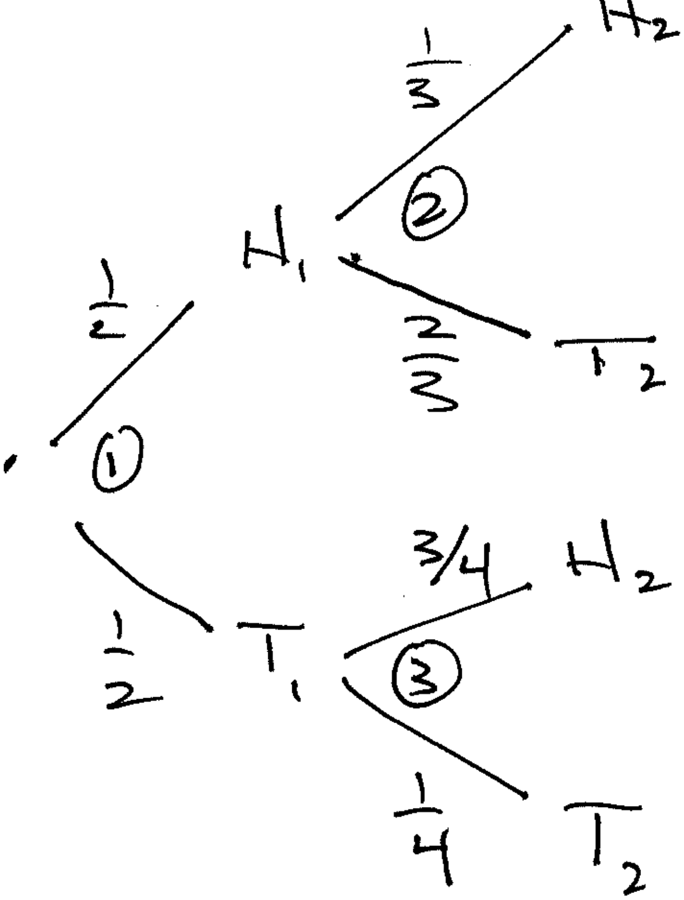
① fair $P(H) = P(T) = \frac{1}{2}$

② weighted $P(H) = \frac{1}{3}$, $P(T) = \frac{2}{3}$

③ " $P(H) = \frac{3}{4}$, $P(T) = \frac{1}{4}$

2 flips:

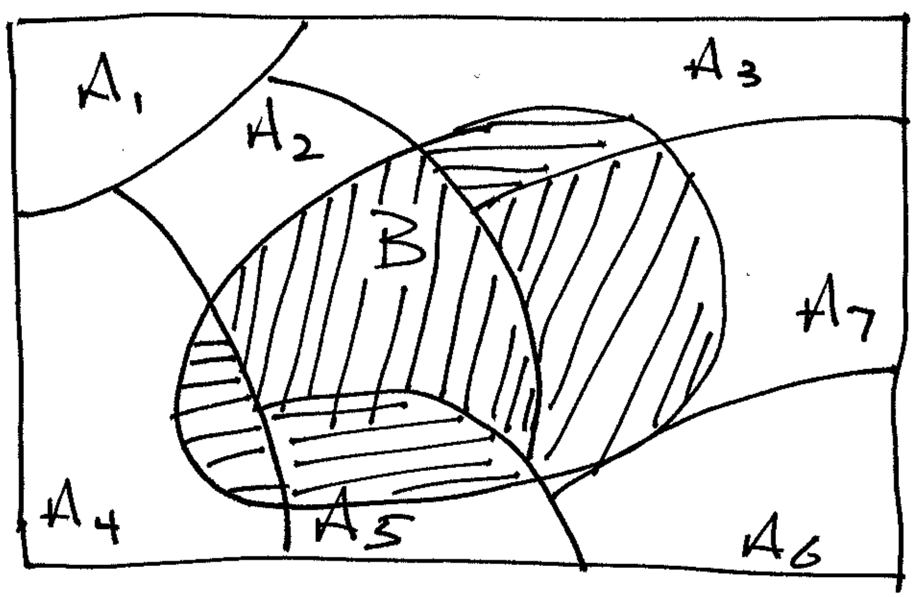
1st flip (1), if head 2nd flip (2), if tail 2nd flip (3)



outcomes	Prob.
HH	$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$
HT	$\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$
TH	$\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$
TT	$\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

1.4 Total Probability, Bayes Rule

Let $A_1, A_2, \ldots, A_n$ a Partition of Ω (i.e. (1) $A_i \cap A_j = \phi$ if $i \neq j$, (2) $A_1 \cup A_2 \cup \cdots \cup A_n = \Omega$).



Ω

Suppose $P(A_i) > 0$ for $i = 1, \ldots, n$.

Let $B \subseteq \Omega$

Total Probability Theorem

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$

$$= P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + \cdots\cdots + P(A_n) \cdot P(B|A_n)$$

Ex. Roll a (6-sided, fair) dice. if result $\in \{1, 2, 3\}$, roll again, otherwise stop. what is Prob. that sum of all rolls is ≥ 5.